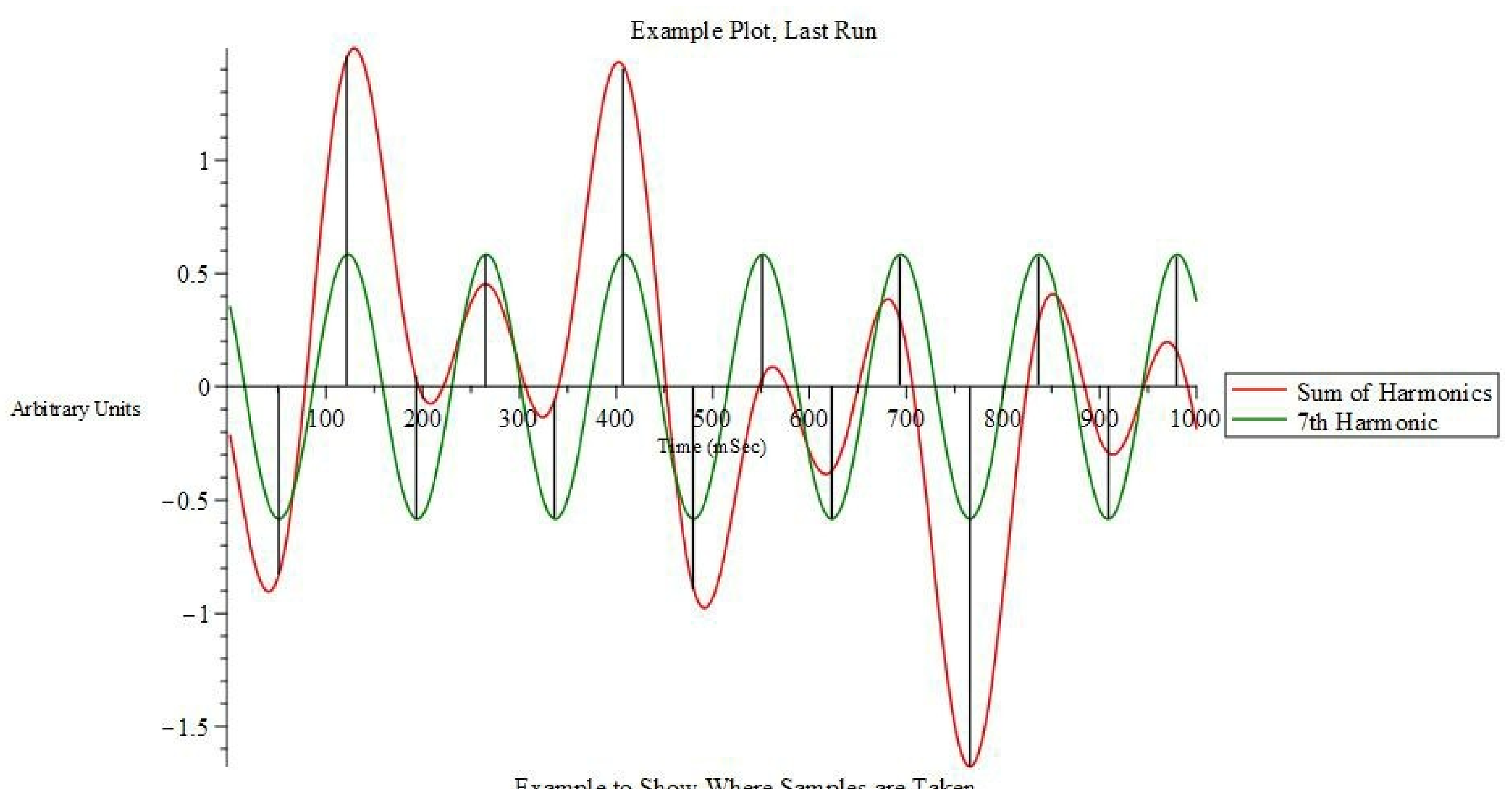
Highest-Harmonic Samples Algorithm Edition, 26 July 2020

Hi all,

Similar to the previous statistical evaluation of randomly selected harmonics with random phases and amplitudes, presented here are some results to evaluate the behavior of an algorithm for estimating the maxima.

The algorithm consists of obtaining the maximum absolute value of an array of calculations that are performed to sample the complex waveform made up of a fundamental and some harmonics with random phases and amplitudes. The samples are picked at the maxima and minima of the highest harmonic, which correspond also to the zeros of the derivative of the function representing the highest harmonic. The plot immediately below illustrates how the values are sampled for the case of fundamental plus two harmonics, specifically the fourth and seventh:



Example to Show Where Samples are Taken

Consider the sine wave:

$$\alpha_h \cdot \sin(\omega_h \cdot t + \phi_h)$$

whose derivative is

$$\alpha_h \cdot \omega_h \cdot \cos(\omega_h \cdot t + \phi_h)$$

For the purposes of this algorithm, we can consider the zeros for this derivative located at

$$\frac{(2 \cdot n - 1) \cdot \pi - 2 \cdot \phi_h}{2 \cdot \omega_h}$$

where n goes from 1 to the maximum $2 \cdot h$ for the harmonic "h." In terms of the fundamental, this can be rewritten as

$$\frac{(2 \cdot n - 1) \cdot \pi - 2 \cdot \phi_h}{2 \cdot h \cdot \omega_1}$$

For the purposes of this note, $\omega_1 = 1$ so the expression simplifies to

$$\frac{(2 \cdot n - 1) \cdot \pi - 2 \cdot \phi_h}{2 \cdot h}$$

where "h" is the highest harmonic, for example $h = 7$ for the case illustrated above, resulting in 14 estimates of the maximum absolute value of the complex waveform. The first estimate is at

$$\frac{\pi - 2 \cdot \phi_h}{14}$$

while the last estimate is at

$$\frac{27 \cdot \pi - 2 \cdot \phi_h}{14}$$

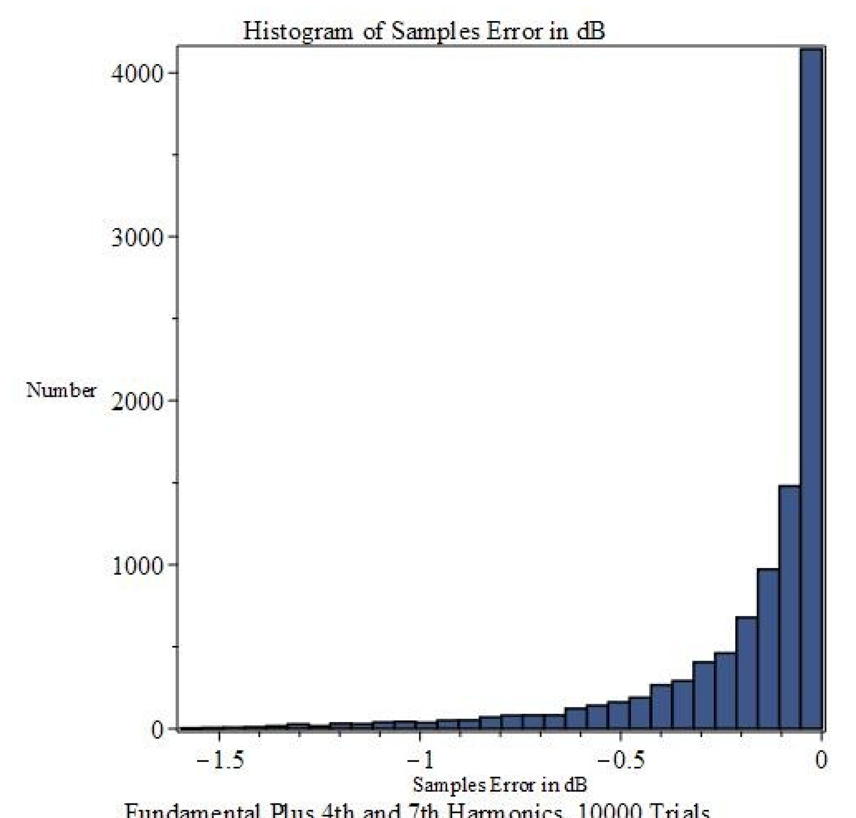
Because ϕ_h can range from 0 to $2 \cdot \pi$, the first estimate may be negative, but this isn't of consequence because the complex waveform is periodic with the same period as the fundamental. In a manner of speaking, the wraparound is handled automatically, thanks to this periodicity. As a sanity check, we can consider the case of $\phi_h = 0$. The first and last estimates are taken at

$$\frac{\pi}{14} \text{ and } \frac{27 \cdot \pi}{14}$$

respectively as expected.

What is shown below using histograms is how well or poorly the maxima of the absolute values of the complex waveform match the actual maxima of the absolute values as estimated by calculating values of the complex waveform at 1,000 evenly spaced samples.

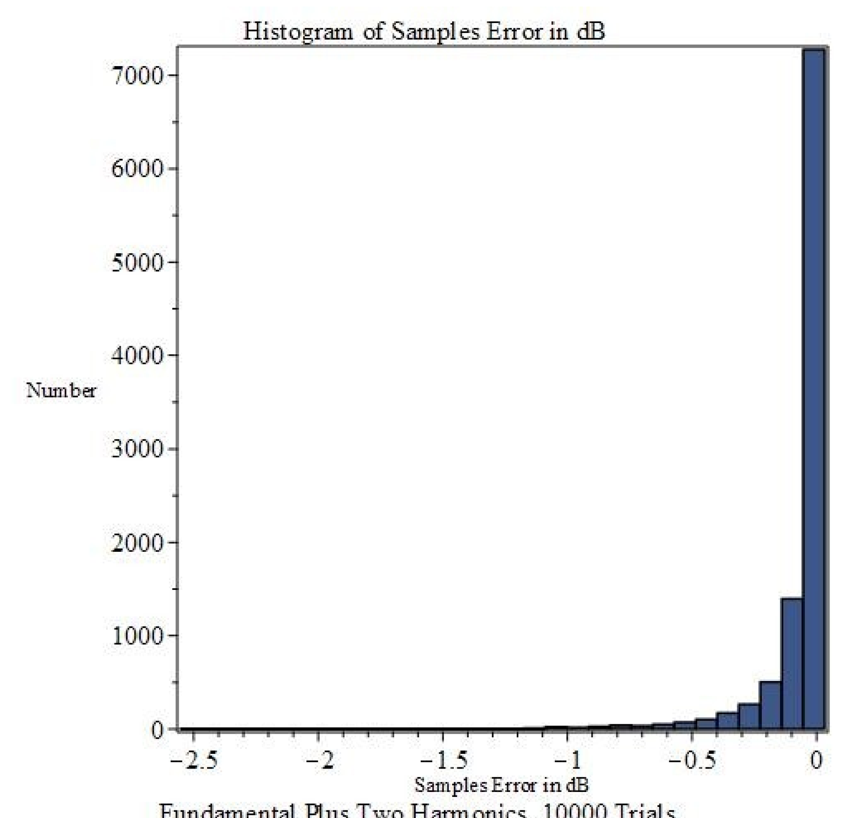
As before, I first simulated 10,000 variations of the problem described by the OP, namely a fundamental and two harmonics, specifically 4x and 7x. Again, the phases were sampled from a uniform distribution between 0 and $2 \cdot \pi$, and the amplitudes were sampled from a uniform distribution between 0.1 and 1.0. The values for 1,000 evenly spaced points were calculated for each waveform before summation. As described above, the maximum of the absolute values at the sampled locations was extracted and compared to the sum of the amplitudes. The "error" was calculated in dB. The result was that the median "error" was about -0.09 dB.



Fundamental Plus 4th and 7th Harmonics, 10000 Trials

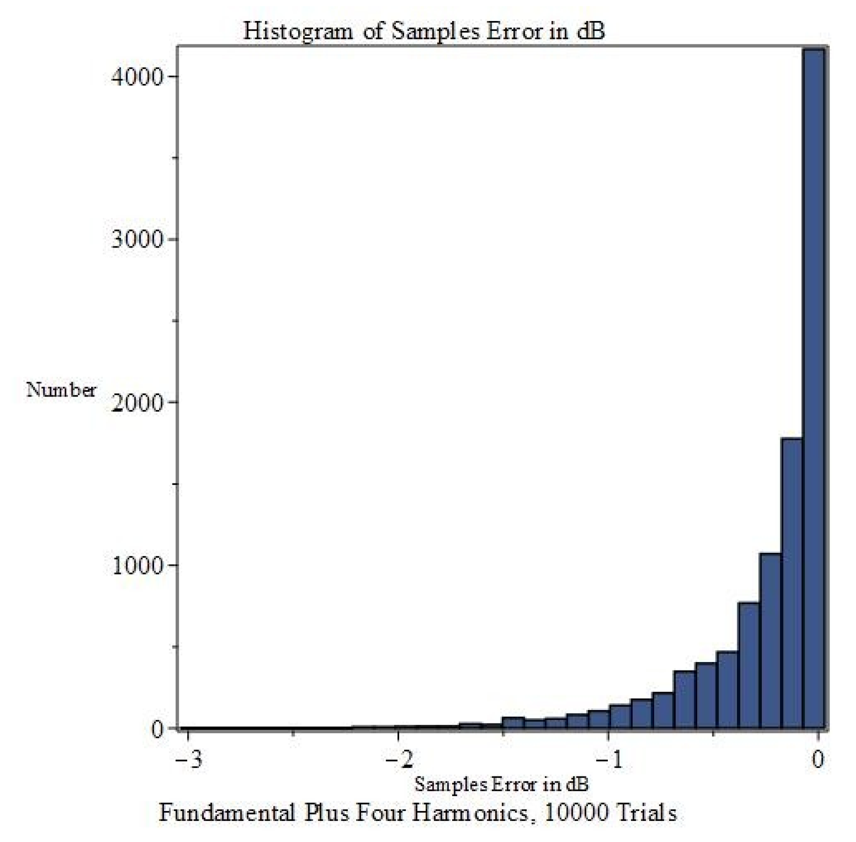
Next I simulated 10,000 variations of a modified version of the problem above to determine the effect of allowing two harmonics to be chosen randomly from a limited set of values from 2x to 32x. (The OP posted 2x to 128x but did not specify the range of the number of harmonics, nor how it might vary.) The values for the harmonics for each trial were sampled from a uniform distribution from 2 to 32 inclusive. The result was a median "error" of -0.011 dB. For this case, for 50% of the time there will be excellent agreement with infrequent exceptions that, although much larger, are only about -2.5 dB at the largest.

If these larger errors are too much to bear, this method could be supplemented by another algorithm such as the "Golden Section Search" as described in (*) below, for example using the zeros to the left and right to bracket the maximum, assuming that the algorithm described here actually found the correct section. Alternatively, the second-highest harmonic could additionally be used to sample the complex waveform in the same manner as described here. Although I have not investigated it, the larger errors may be caused whenever the amplitude of the highest harmonic is low compared to those of the others, reducing its influence on the complex waveform, hence upon its reliability to point to the correct section nearest to the maximum absolute value.

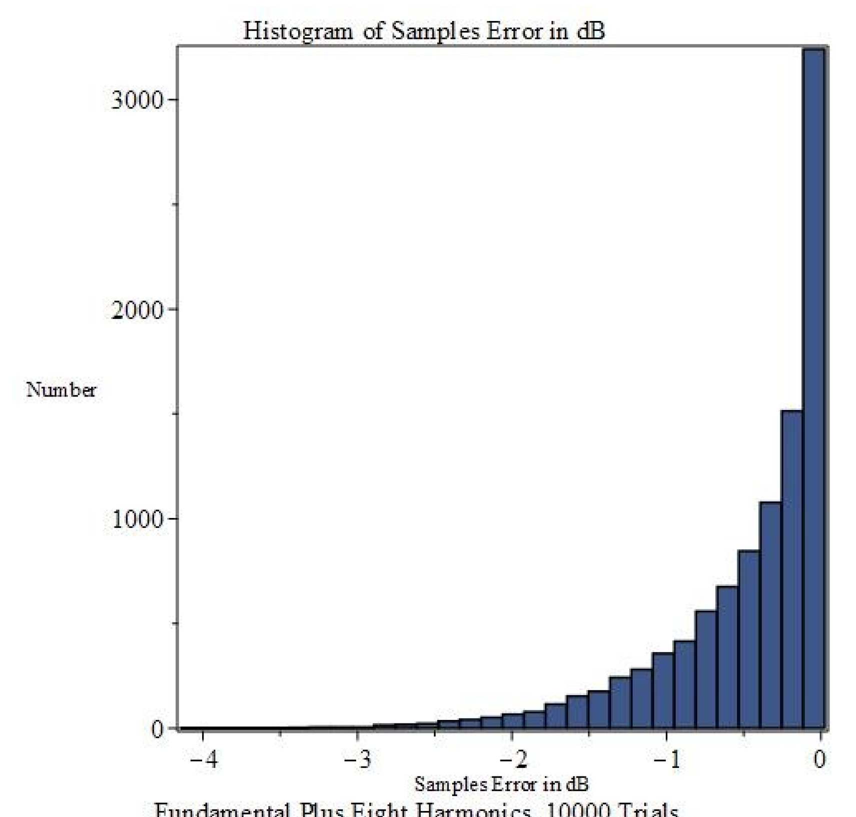


Fundamental Plus Two Harmonics, 10000 Trials

I repeated the same type of simulation with the fundamental plus four harmonics from 2x to 32x, with the result that the median "error" was -0.11 dB, then fundamental plus eight harmonics with the result of -0.28 dB. Here again is displayed a long tail indicating infrequent, larger values.

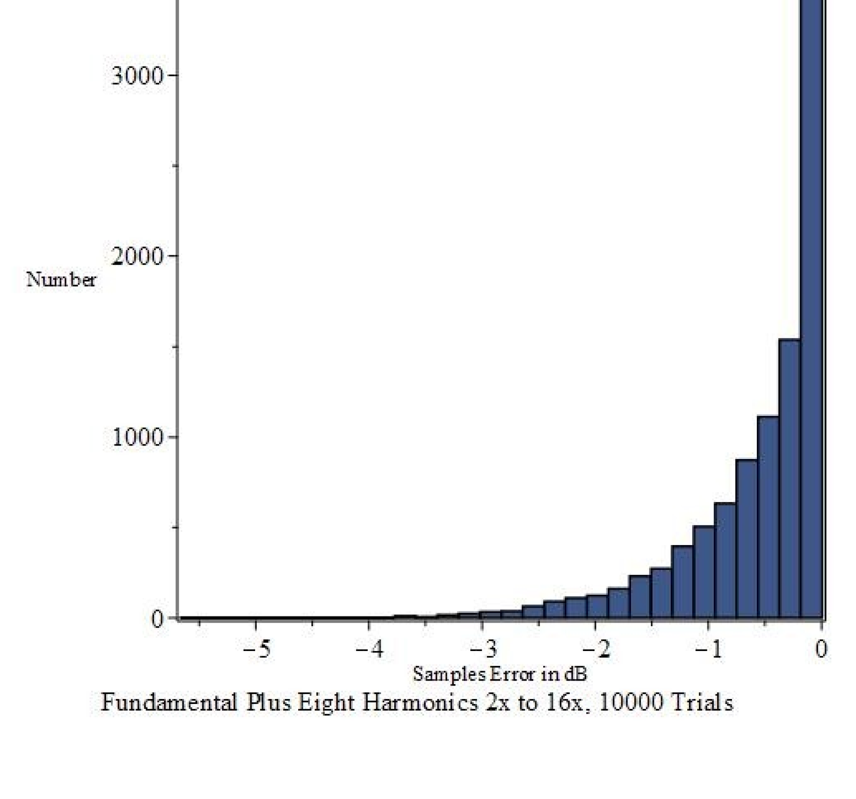


Fundamental Plus Four Harmonics, 10000 Trials

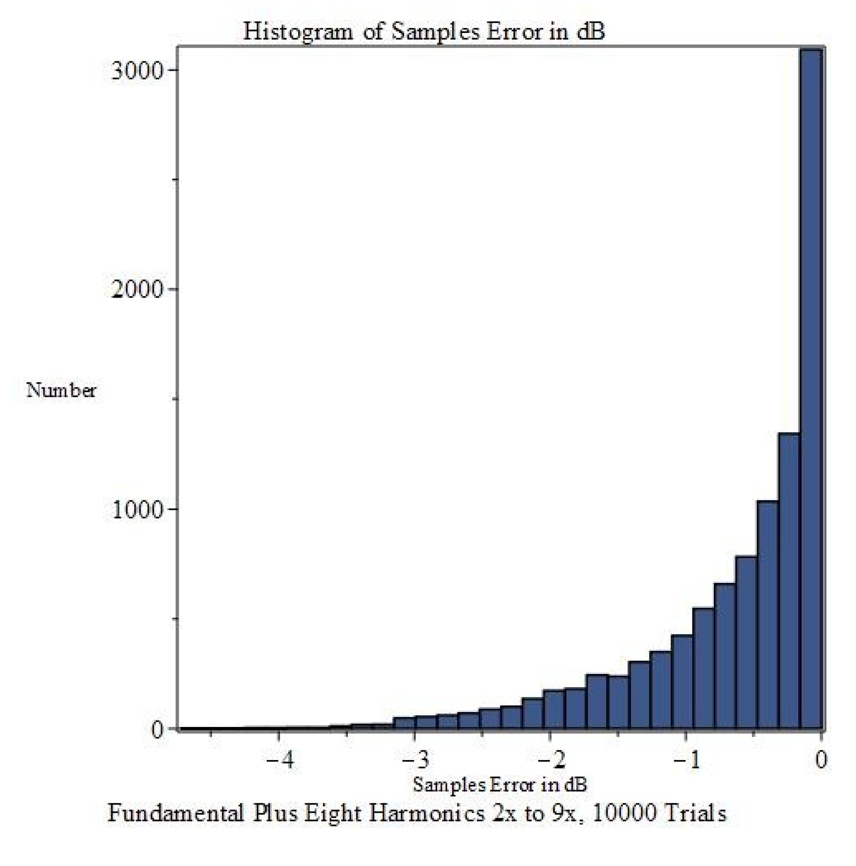


Fundamental Plus Eight Harmonics, 10000 Trials

To determine what a more dense set of harmonics might produce, I then limited the range of harmonics from 2x to 16x and repeated the calculations for fundamental plus eight harmonics with the result that the median "error" was -0.33 dB. The next result was for 2x to 9x for a full set of harmonics in that range with randomly varying phase and amplitude and was -0.39 dB for the median "error."

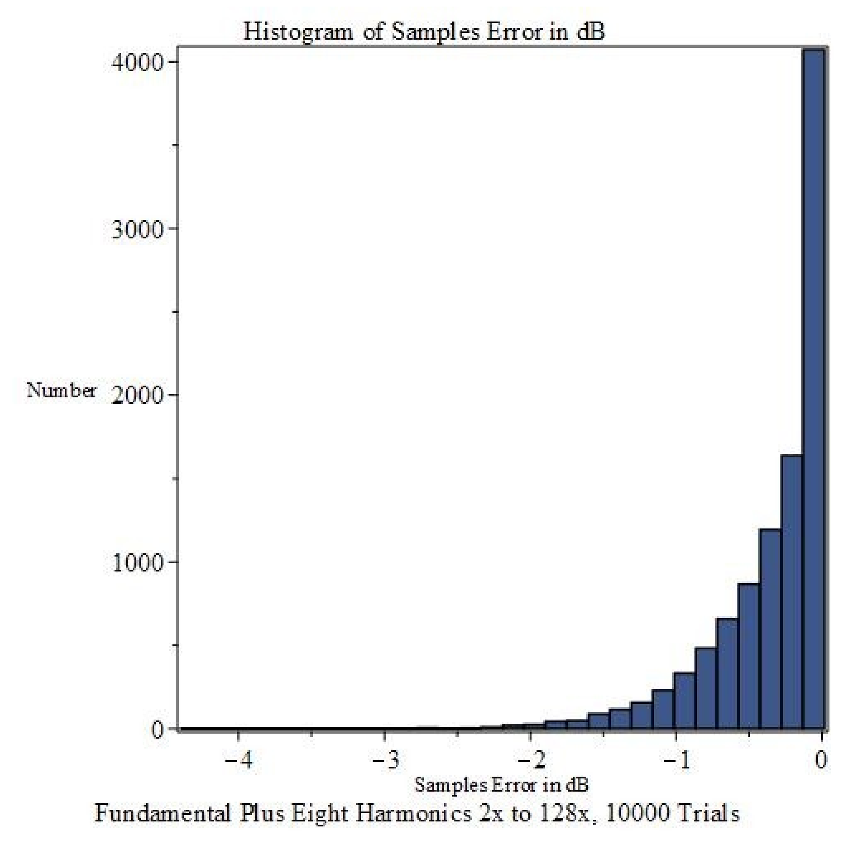


Fundamental Plus Eight Harmonics 2x to 16x, 10000 Trials



Fundamental Plus Eight Harmonics 2x to 9x, 10000 Trials

Finally, with a range of harmonics from 2x to 128x, I looked at the fundamental plus eight harmonics with the result of -0.21 dB for the median "error." However, with only 1,000 samples used to estimate the true maximum of the complex waveform, the initial result was skewed by the fact that with very high harmonics, there is a chance that the sampling will find a better result than the 1,000 samples of the complex waveform to which it is compared. So 5,000 samples were used instead to improve the estimate of the error.



Fundamental Plus Eight Harmonics 2x to 128x, 10000 Trials

Regards,
Dave Clark

(*) William Press *et al.* *Numerical Recipes: The Art of Scientific Computing*. Second Edition, Cambridge University Press, 1992, pages 397 – 402.