The Monty Hall Problem: Probably Not the Last Words, But I Wish It Were So

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This document is a leisurely stroll through many of the technical issues of the so-called "Monty Hall problem." It's intended for a general audience, but having some background in probability may help. Having some background may also hinder understanding, unfortunately. Many students of probability and statistics misconstrue what they studied in such courses and misapply methods, misquote citations, etc. The Monty Hall problem has a particularly sordid history of misunderstandings, far beyond those that are well documented. This document ventures into this territory. Below are some definitions and claims, then a few selected but illustrative myths to correct false claims or to demonstrate just how deep certain misunderstandings have plunged.

Although some readers may have preferred equations, I have kept them out of this document because many people who would avoid reading them are perfectly capable of understanding the words representing them and the points being made.

What I shall call the "actual Monty Hall problem" in this document takes the form [Selvin] [vos Savant]:

In the one and only instance described, a game player had three choices, only one of which led to a favorable outcome. He/she made a choice. The host of the game, who knew which choice led to the favorable outcome, revealed one of the two non-selected choices, and it was unfavorable. A switch was then offered so that the game player may choose the remaining non-selected choice instead of the original choice. Is it to the game player's advantage to switch to the remaining non-selected choice?

It should be noted that the name "Monty Hall problem" is a misnomer because, according to the real Monty Hall, the show he hosted didn't work like this. In 2008, John Tierney, who interviewed Monty Hall in 1991 wrote, "Here's how Monty's deal works, in the math problem, anyway. (On the real show it was a bit messier.)" [Tierney]. Actually, it was a lot messier as anyone can see by watching old episodes.

The first revised version of the Monty Hall problem (MHR1) takes the same form as the actual Monty Hall problem above with three explicit additional conditions in the problem statement: 1) For all instances the placement of the favorable outcome must be random. 2) For all instances of the game, the host is required to reveal a non-selected choice that is unfavorable and must always offer the chance to switch. 3) If the game player initially selects the favorable choice, the host must randomly pick one of the two unfavorable choices to reveal.

MHR1, thanks to the severe restrictions, reduces to a trivial question, for example: For a prize randomly distributed in three places, would you rather have any one choice or any combination of two choices?

The importance of condition 1) for MHR1: If conditions 2) and 3) are met, but the host always puts the favorable outcome in the first position, for example knowing from the history of the game that every game player is 50% likely to initially choose the first position every time they play, then the results are not that switching doubles the chances of winning as for MHR1, but rather that the probability of winning is 50% for both switching and not switching. In this case, there is no advantage or disadvantage for always switching. Furthermore, if the game player knows that the favorable outcome is always placed in the first position, he/she will can make a favorable decision 100% of the time, even if his/her initial choice is forced to be random. This is but one simple example of many that would be problematic if condition 1) is not met. The placement probability must be strictly equal for each position for MHR1.

The importance of condition 2) for MHR1: If conditions 1) and 3) are met, but the host reveals an unfavorable outcome and offers to let the game player switch only when the game player selects the choice with the favorable outcome, then the game player will always lose when switching. In other words, the probability of winning would be 0% for always switching. This is but one example of many that would be problematic without condition 2).

The importance of condition 3) for MHR1: If conditions 1) and 2) are met, but a) the game player always selects choice A where A is fixed at 1, 2, or 3, and if b) the game host reveals the outcome of choice B whenever it is consistent with the problem statement to do so where B is fixed at 1, 2, or 3 but not the same as A, and if c) the remaining choice is called C, then because of random placement of the favorable outcome, two-thirds of the time the favorable outcome will be placed in choice A or C, with A or C equally likely, and the game host will reveal that choice B leads to an unfavorable outcome. Therefore, if the game host opens choice B, then the probabilities for obtaining the favorable outcome for switching and not switching are both 50%, and there is no advantage for switching. If the game host does not reveal the outcome of choice B, then choice B contains the favorable outcome and the probability for obtaining it for switching is 100%. The average probability of winning for always switching is two-thirds of 50% plus one-third of 100% or two-thirds.

An example would undoubtedly help here: Say the player always picks 1, and the host always reveals 2 unless it contains the favorable outcome. Under these conditions, whenever the host reveals 3, then the probability that 2 contains the favorable outcome is 100%. If the host reveals 2, then the probability that 3 contains the favorable outcome is 50%, and there is no advantage to switching. The complicated description in the previous paragraph is a generalization of this specific example using A, B, and C.

It is quite ironic that without condition 3), even with the other conditions in place, it is possible for the game player's and game host's strategies to create a situation wherein astute observers can detect the strategies and use them to calculate the probabilities for individual games. If knowledge is defined as true belief that is not the result of luck, then these astute observers know whether or not there is any advantage for switching for each game right at the time of the problem statement that this question is asked. They can give a sequence of answers for one game after another at this critical time, for example: "No, No, No, Yes, No, Yes, …." Two-thirds of the time there will be no advantage to switching, precisely what many people over the years have claimed is not possible, especially when conditions 1) and 2) are met. Condition 3) is necessary to ensure that each and every game has a probability of obtaining the favorable outcome of two-thirds for always switching, and not simply that there is an average probability, over many games, of two-thirds.

Experiments usually show that humans are not good at creating random sequences [Figurska et al.], so it's doubtful that conditions 1) and 3) could ever actually be satisfied by human beings without some sort of assistance.

The second revised version of the Monty Hall problem (MHR2) also takes the same form as the actual Monty Hall problem above, but has two explicit additional conditions in the problem statement: 1) Same as for MHR1. 2) In all instances, the host is required to open a door, any of the three doors, but does so randomly.

Considering only the situations matching the problem statement, MHR2 reduces to the trivial question: For a prize randomly distributed in two places, would you rather have this one or that one?

The importance of condition 1) for MHR2: If condition 2) is met, but the host always puts the favorable outcome in the first position, for example knowing from the history of the game that the every game player is 50% likely to initially choose the first position every time they play, then the results are not that the chances of winning are 50% for both switching and not switching as for MHR2, but rather that the probability of winning is one-third for switching if and when the situation described in the problem statement is reached. In this case, there is a disadvantage for always switching. This one is tricky to work out, but one way to do it is to substitute one-half for some of the values for one-third or two-thirds in the following two paragraphs. The results should be that three-sixths (or one-half) of the time, the problem statement conditions will not be met, while the other half of the time they will be met. Two-sixths of the time it is not favorable to switch and the problem statement conditions are met while one-sixth of the time it is favorable to switch and the conditions are met. The odds are two to one against switching when the conditions are met.

The importance of condition 2) for MHR2 can perhaps be best shown by demonstrating how the game host's random choice of revealing an outcome produces no advantage for switching with MHR2. Assume that conditions 1) and 2) are both met for MHR2. The randomness of the selection of which door the game host opens results in instances of the game player losing and winning before being given the chance to switch, but these instances do not match the problem statement. Of these instances that do not match, the possibilities are: 1) game player selects the choice with favorable outcome, and the game host opens it due to random selection of choice; 2) game player selects one of two choices with unfavorable outcomes with total probability of two-thirds for both choices due to random placement, and the game host opens it due to random selection; 3) game player selects one of two choices with unfavorable outcomes with total probability of two-thirds for both choices due to random placement, and the game host opens the choice with the favorable outcome due to random selection. Out of all instances, 1) occurs one-third of the time times one-third of the time or one time out of nine, while 2) occurs two-thirds of the time times one-third of the time or two times out of nine, and 3) occurs two-thirds of the time times one-third of the time or two times out of nine. Adding these results gives five times out of nine that will not match the problem statement.

Continuing the discussion of the previous paragraph, the remaining instances which do match the problem statement are 1) game player selects the choice with favorable outcome, and the game host opens one of the two choices with unfavorable outcomes with equal probability for both due to random selection; 2) game player selects one of two choices with unfavorable outcomes with total probability of two-thirds for both choices due to random placement, and the game host opens the other choice with an unfavorable outcome, as a consequence of random selection. Of all instances, 1) occurs one-third of the time times two-thirds of the time or two times out of nine, while 2) occurs two-thirds of the time times one-third of the time, also two times out of nine so that the probability of obtaining the favorable outcome for both switching and not switching is 50% because these two probabilities are equal.

We have five times out of nine that do not match the problem statement and four times out of nine that do match, so we have accounted for all cases. The probability of obtaining the favorable outcome for switching and not switching for situations that match the problem statement is 50%; there is no advantage to switching.

Note that MHR1 and MHR2 are both consistent with the actual Monty Hall problem statement, but neither one is equivalent to it. The actual Monty Hall problem cannot be reduced to a trivial question as can MHR1 and MHR2. The single described instance of this actual problem is consistent with an infinite number of revised problems in addition to MHR1 and MHR2.

Note also that a key difference between MHR1 and MHR2 is that when a non-selected choice is revealed as having an unfavorable outcome, the probability associated with that choice is distributed differently among the remaining choices. In the case of MHR2, that probability is divided between the currently selected choice and the other non-selected choice: one-third divided by two or one-sixth is added to each choice to increase the probability from one-third to one-half for both choices. In the case of MHR1, that probability is not divided with the currently selected choice which retains its previous probability: one-third is added to the non-selected choice bringing it to two-thirds, while the currently selected choice remains at one-third.

Most discussions of the so-called "Monty Hall" problem pose what I am calling the "actual Monty Hall problem," then proceed to offer a solution for MHR1, usually without mentioning any of the required additional conditions in the problem statement [Selvin] [vos Savant] [Note 5] [Note 6]. In other words, very often authors actually do not solve the problem they posed. Here is a fairly recent rendition by mathematics professor and author John Allen Paulos in Scientific American [Paulos], published 36 years after Selvin's article:

"A guest on the show has to choose among three doors, behind one of which is a prize. The guest states his choice, and the host opens one of the two remaining closed doors, always being careful that it is one behind which there is no prize. Should the guest switch to the remaining closed door?"

Note that Paulos, in addition to not stating conditions 1) and 3), did not actually state condition 2). This problem statement could be interpreted to mean that when and if the host opens one of the two remaining doors, he is always careful to open one such as not to reveal a prize. So is the host forbidden to open the door chosen by the guest or not? "Always being careful" conveys a different meaning than "always opens." Instead Paulos should have written something like, "… and the host always opens one of the two remaining doors, never revealing a prize."

A few people solve the "second revised" Monty Hall problem or MHR2, usually without mentioning the required additional conditions in the problem statement. They also usually do not solve the problem posed.

The actual Monty Hall problem could be said to have a solution for any particular instance of the problem, but there is not enough information provided to find it. In this sense, it could be said informally that there really is no solution to the actual Monty Hall problem posed by Steve Selvin, Marilyn vos Savant, etc. On the other hand, it could also be said that the problem has a potentially infinite number of probabilities for winning between 0% and 100%, leading to indecisiveness about whether or not there is an advantage to always switching. Some say that the actual Monty Hall problem is "ambiguous" as if it was somehow illegitimate; but it's no more ambiguous or less legitimate than many questions in life, and in science and engineering. We are very often faced with "ambiguous" problems that require a decision whether we like it or not. We very often do not have the opportunity to change the question to one that is more convenient and more to our liking. The latest data may show that our vehicle approaching Mars isn't where we thought it was. Should we slow the rate of descent or not?

As many of us are well aware, there is and always has been a great deal of controversy about the actual Monty Hall problem. Controversies usually involve purported solutions to this problem which are inconsistent with each other. For example, a lot of acrimony has developed between those who felt that the solution to MHR1 was the one and only solution to the actual Monty Hall problem and others who felt that the solution to MHR2 was the one and only solution to the actual Monty Hall problem. While either side may be correct about a particular instance of the game, neither side is correct in general [Note 4]. Controversy has also developed about claims that the solution for MHR2 was actually also the solution for MHR1, claims which are provably false. Always switching is advantageous in the case of MHR1 but not for MHR2. Never switching is disadvantageous in the case of MHR1 but not for MHR2.

MHR1 has become so widely accepted as the actual Monty Hall problem that it is sometimes called the "standard" or "classic" Monty Hall problem. Unfortunately, the standard way to state this "standard" version is to err and state the actual Monty Hall problem. Occasionally this "standard" version appears with only the condition that the host must always show a non-selected and unfavorable choice and offer the switch, condition 2) above. More commonly, informal arguments are made in the purported solution that condition 2) must be true; but informal arguments are not generally acceptable in mathematics. These arguments are, for example, "Surely the game host wouldn't …"; "You wouldn't … would you?"; etc. In mathematics, it generally doesn't matter what you yourself would do, rather what it is possible to do.

Almost all of the discussion that one will find on the Internet is technically incorrect because the solution to MHR1 is offered as the solution to the actual Monty Hall problem or to a version of MHR1 without the required conditions 1) and 3) explicitly stated. The same is true for many textbooks, including college-level probability and statistics textbooks which discuss similar problems such as the three-shells game which offers the opportunity to switch choices.

Many people have difficulty understanding MHR1. Typically these people argue that there is no advantage to switching choices, either one resulting in a 50% chance of obtaining the favorable outcome. In other words, they insist on applying the solution for MHR2 to the problem of MHR1 even when conditions 1), 2), and 3) above for MHR1 are explicitly provided to them. This is unfortunate because MHR2 is no more and no less valid an interpretation of the actual Monty Hall problem than is MHR1 or any one of the other infinite number of possible interpretations. The solutions for MHR1 and MHR2 are equally INVALID as complete solutions to the actual Monty Hall problem.

There is a cottage industry creating books, magazine articles, and so on full of various types of puzzles, and the Monty Hall problem seems to be fairly popular for inclusion in works of this kind, especially in collections of so-called “brain teasers.” It’s possible that maintaining this problem as a brain teaser is the primary reason why MHR1 has become the “standard” Monty Hall problem. If MHR2 had been accepted as the “standard” one, then it almost certainly would have been dropped as a brain teaser because the “standard” answer would be that it doesn’t matter whether you switch or not, something that many readers would immediately conclude anyway. “Where is the tease?”

The desire to maintain this problem as a “brain teaser” could also explain why authors are so sloppy about including the three required explicit additional conditions for MHR1 in their problem statements. Including these conditions nearly doubles the length of the problem statement and probably would put a lot of readers off before they finished reading it due to the dry mathematical nature of the conditions, especially taken together. The fact that the lengthier version can mechanically be reduced to a trivial question (one choice or any two?) that has an obvious answer (two) would probably put off just about everyone else. “Duh!” But of course, there are many authors who don’t seem to understand that all three conditions must be explicitly stated for the problem to be unambiguous. Undoubtedly this leads to some readers feeling that they are being misled, and indeed they are. In these cases, the problem becomes more of a swindle than a brain teaser.

There are many myths surrounding the Monty Hall problem in all its versions. Here are a few with comments, sometimes quite long comments.

Myth: MHR1 is a logic problem, not a probability problem. If it was a logic problem, then the answer would be something like: You chose the curtain with the car behind it, don’t switch. Logic problems have definite conclusions, not probabilistic ones. They don’t have expected payoffs like one-third of a car or two-thirds of a car. The definite conclusion of a logic problem is entailed in the problem description, in the premise or premises. The fact that MHR1 uses logic to solve it does not make it a logic problem. In logic problems, the taller native always tells the truth or always lies; the guy who left the room had on a red hat or a white hat, not a pink one.

Myth: The actual Monty Hall problem is a probability problem. This is not true, either. MHR1, on the other hand, is a very trivial game theory problem that can be treated as a probability problem with very little or no loss of generality because it leads to two definite probabilities of winning at the time the question about switching is asked. The host’s strategy is completely defined. The game player’s two strategies are completely defined: game player has not yet made his/her last move, but once they do, the probability of winning is determined exactly.

The actual Monty Hall problem is a game theory problem. Depending upon the strategy of the host, there are different sets of probabilities for winning, and which set of probabilities is the correct set is unknown because the chosen strategy of the host is not specified in the problem statement, so the probability of winning ranges, from zero to one, over the various sets of probabilities. It looks to many people like a probability problem, but it’s not, because the information is too incomplete for that.

MHR1, considered as a probability problem, does contain logic, but that doesn't make it a logic problem. It is beyond a mere logic problem. The actual problem, a game theory problem, does contain probabilities and logic, but it is beyond a mere probability or logic problem.

Myth: There are two games. No, not in the sometimes claimed sense that one game uses three curtains, and the second game starts at the time the first game ends when the host asks the game player if he/she wants to switch. There is but one game in the problem statement for both the actual problem and MHR1. This because there is no payoff at the point prior to the host asking about switching. There is instead another turn taken by the host. The fact that this turn may take the game player by surprise does not end one game and start another one.

Myth: There are two games because the rules are changed from the perspective of the game player when the switch is offered. Changing the rules does not start a new game. Changing the rules is merely a move in what is sometimes called a "multi-stage game." The actual rules are established elsewhere and are not changed. In both the actual and MHR1s, the rules are set by the host before the game even starts, for example. He/she allows or requires him/herself to offer the switch of choices to the game player. So the rules actually are not changed. Rules are often apparently changed in real games in real life. A parent forbids a teenager to leave the house. He/she is out the door when the parent suddenly says, "Well OK, but be back before midnight." The actual rule is obviously something different than "the teenager may not leave the house when forbidden by the parent." In the actual Monty Hall problem and MHR1, the reader is fully apprised of what happened and cannot complain that the rules have been changed, therefore that there are two games. The reader knows full well that the offer to switch either may (actual) or must (MHR1) be offered when the problems are correctly posed.

Myth: The 100-, 1000-, 1,000,000-door problem results in an overwhelming advantage for switching. Alternative Myth: The 100-, 1000-, 1,000,000-door problem results in a 50/50 chance for switching (no benefit for switching). See [Note 3] for an example problem statement. Although psychologically powerful, the arguments presented for both of these claims are invalid when applied to the actual Monty Hall problem, regardless of the fact that either conclusion may be correct for a particular case. There is too much ambiguity to be certain. Furthermore, it may also be that switching leads to an overwhelming disadvantage.

Consider this psychologically powerful counter-example I developed before seeing similar ones developed by others, a million-ticket lottery in which you have purchased a ticket. On the night of the drawing, no announcement is made. Several days later, lottery officials show up at your door (having obtained a record of your address that you provided when you purchased the ticket). They inform you that they have eliminated all but two of the tickets from being the winner of the super jackpot, your ticket and the ticket of one other person. According to a number of people who have discussed the Monty Hall problem, you should most definitely switch to the other ticket. You are in luck because also according to these same people, that other person should switch to your ticket. Swapping tickets is consistent with this, but, according to the argument offered by these people, swapping will dramatically increase both of your chances of winning. Of course, this is not possible, despite a few claims to the contrary. Either the chances are 50/50 or else one of you has an advantage over the other. If one of you is "special" (for example, never would have had your ticket eliminated AND you probably don't have the winning ticket), the "special" one has the advantage while the other one suffers a disadvantage for switching (for the same example, because they probably do have the winning ticket and would probably swap it away). If all eliminations were due purely to chance, then it's 50/50. There isn't any way to state definitively which of these is the case for any particular instance of this lottery game, or somewhere in-between (for example, narrowed down to ten possible winning tickets), so the actual probabilities cannot be calculated.

Similarly, with three players of the actual Monty Hall problem, one player MAY (or may not) be eliminated along with revelation of an unfavorable outcome. It may be that the odds are 50/50 for the remaining two players (for example, random elimination), but it may also be that one player has the advantage over the other and that the odds of either one winning can be anywhere between 0 and 1, with the sum of odds always adding to 1. Similar to before, if one player would never have been eliminated under any circumstances and probably did not choose the favorable outcome, they are "special" and have the advantage. Nevertheless, for the actual Monty Hall problem, it's still impossible to calculate the odds because the host's strategy is not known. It could be, for example, that Monty offers the switch only when someone he doesn't like has chosen the favorable outcome. He would have liked to have eliminated them immediately, but he can't because they chose the favorable outcome, so he offers them the chance to switch and self-eliminate. The game player should not switch. Or it could be that Monty offers the switch only when someone he does like has chosen the unfavorable outcome. He wants to keep them around, so he eliminates one of the others, then offers this player the chance to switch. He/she should switch.

The psychological trick that has been played (probably unwittingly) on many people who accepted such arguments from Martin Gardner, Marilyn vos Savant, and many others over the years is that a single game player is the only player, so they are "special" but don't feel "special," being the ONLY player. They probably also don't understand the implications of being "special." At the beginning of MHR1, for example, the game player probably did not pick the choice with a favorable outcome. He/she was not eliminated and would never have been eliminated due to condition 2). Thus there is an advantage to switching. For the 100-, 1,000-, 1,000,000-door problem with one game player, the same trick has been played. Again the game player doesn't feel "special" because he/she is the only one. But this game player almost certainly did not initially pick the correct door, and not only were they not eliminated, they would never have been eliminated. For the actual Monty Hall problem, the game player probably did not make the right choice for a favorable outcome, but it is completely unknown whether or not they could possibly be immediately eliminated. It is not known whether they are "special," or the opposite of "special," or somewhere in-between.

Another way to critically examine the 100-, 1,000-, 1,000,000-door problem and Marilyn vos Savant's advice, probability calculations, etc. is to consider what would happen if the doors were opened in a slightly different way. Let's add an intervening step which is not prohibited by vos Savant: Suppose the game host opens up all doors of a 100-door game except for doors 55 and 77 and the initially chosen door of the game player. At the outset of the game, each door has a one percent chance of having the favorable outcome if we use condition 1) of MHR1 or MHR2. When all the doors except for three are opened, according to vos Savant and others, the probability of the initially chosen door remains at one percent, but the other two doors are now at 49.5 percent each, having shared the 97% probability for the 97 doors opened plus their own initial one percent. According to vos Savant's advice about switching to higher odds when they develop, the game player should switch to either of door 55 or 77 if asked. Now the game host opens one of the doors not selected, chosen at random according to 3) of MHR1.

First let's say the game player switched to door 77, and the game host opens up door 55. Because the probability of door 77 does not change for MHR1, the probability of 49.5 percent for door 55 is now distributed only to the door that was the initial choice of the game player, bringing the probability for that door to 50.5 percent while that of door 77 remains at 49.5%. In this case, the game player is obligated to switch back to the original choice! And the odds are now nearly 50/50. Now suppose that the game player still switched to door 77, but the game host instead opens up the door originally chosen by the game player. We have now one percent that is awarded to door 55, bringing its probability to 50.5 percent while that of door 77 remains at 49.5%. Again, the odds are very nearly 50/50 and the game player must switch because the odds are slightly higher to do so. In this game, using the advice, logic, and probability calculations of vos Savant and others, the game player must switch twice regardless of which doors are chosen. Perhaps somewhat surprisingly, the door initially chosen by the game player becomes equal in status to the doors skipped over by the game host, regardless of which door the game player initially chose.

Also note that as the number of doors in this game increases, the closer the odds get to 50/50 for the last step of the game. For 1,000,000 doors it's 49.99995 percent versus 50.00005 percent.

Once again we have a rather surprising result: The odds for all doors were initially equal. The last part of the game was a three-choice problem, and yet the odds are nearly 50/50. We're still always supposed to switch, and when asked if we want to switch back to our original choice, we are supposed to do that. According to the original argument with only door 77 skipped over, that is surely a bad idea!

The only way around elevating the status of the door initially chosen is to prevent the game player from switching at the first step of the modified game, or for the game player to refuse to switch at the first step, contradicting the advice to switch to capture the maximum probability at every step of the way, which is based on the reasoning of vos Savant and others. This strategy would keep the odds at the original value of one percent. The only way the game player would agree to do this would be if he/she could be assured, for the case of one million doors with one door opened at a time, that as long as he/she refuses to switch up to 999,997 times and agree only once, and only the last time the question could possibly be asked, then his/her chances of obtaining the favorable outcome could practically be guaranteed. In this one-door-at-a-time game, vos Savant's advice should be ignored up to 99.9999 percent of the time in order for the switch at the last opportunity to work as vos Savant described. Conveniently in her description of the 1,000,000-door problem, she chose to use only one step (the last and crucial one) to immediately reduce the many-many-door problem to a three-door problem.

Myth: The original rendition of the Monty Hall Problem by Steve Selvin states that the host must open a curtain revealing a goat. This is simply not true. To his credit, Selvin did state that the original probability of picking the keys to the car was one-third, consistent with condition 1) of MHR1. He did not state conditions 2) or 3) in the problem statement. The original letter to the editor by Steve Selvin is in a format in which there is a definite separator between the problem being posed and the purported solution to this problem. This separator is the name of the author, his affiliation, and his location. This format is very similar to that of quizzes many of us have seen in newsprint magazines in which answers to questions on a page are printed upside down at the bottom of the page. It's also similar to textbooks which have answers at the back of the book with the separator for a particular set of problems being the remainder of the book after the particular problem set is presented.

In the original article, after his biographical information, and at the beginning of his purported solution, Selvin provides an informal argument that the host of the game show would not open the box containing the keys to the car. This is neither a formal argument that shows how the premise or premises entail the required condition (which isn't possible anyway) nor a declaration that this must be presumed to be a premise of the problem statement. Instead it is essentially a statement of what the author would have done if he were the game host, which has nothing whatsoever to do with the working of the problem; Selvin should have considered all possible strategies allowed by the problem statement he wrote. If he wanted to stop Monty from ever opening the box with the keys, he should have stated so in the problem statement. This error of posing one problem then solving another is, of course, a huge blunder that was unfortunately duplicated by almost everyone ever since, most notably and even more disastrously by Marilyn vos Savant in her Parade Magazine article. Once again, in the original article describing the Monty Hall problem, the author posed one problem and then described the solution to a different problem.

Myth: There are only two *real* choices in MHR1 with no advantage to switching because the third choice is always a throwaway choice that has no effect so can be eliminated before the game starts. This claim is based on a very serious mistake: assuming that the order of the moves by the game player and game host can be interchanged with no effect on the result. The problem with this is that the game player and the game host are not equivalent. The game player is not restricted at all in his/her initial choice in MHR1, but the game host is required to reveal a choice with an unfavorable outcome and must also select a choice that is not that of the game player. If the game player goes first (MHR1), he/she always has three choices. Two-thirds of the time the game host who goes second has one choice left for revealing the unfavorable outcome and two choices left only one-third of the time. On the other hand, if the game host goes first, both the game host and the game player always have two choices available. If the game player goes first (MHR1), two-thirds of the time he/she will initially select a choice with an unfavorable outcome and be offered the chance to switch to a choice with the favorable outcome, but if the game host goes first eliminating one choice, the game player will initially choose a choice with an unfavorable outcome and be offered the chance to switch to the favorable outcome only half the time.

Although it may be hard to see at first, the effect of the third choice for the game player is to decrease the initial probability of winning, compared to only two choices, while simultaneously increasing the chance of losing. When one choice is subsequently eliminated and a switch is offered, the disadvantage and advantage can be exchanged, resulting in a kind of reversal of roles. This myth is an attempt to short-circuit all of the stuff that is critically important for creating the initial disadvantage which can then be turned into an advantage, without changing the problem. It fails to do so; the problem is changed.

Myth: Informal arguments can be used to justify the claim that the actual problem is identical to MHR1. In other words, problem statements that leave out the requirement that the host always reveal a goat should always be interpreted such as if this requirement was explicitly included. These informal arguments are things like: "Surely Monty wouldn't insist on revealing a goat if the game player chose it!" "It would be unethical to offer a switch to cause the game player to only lose the game!" "Monty wants to increase the likelihood of winning so that the show becomes more popular!" These claims are nothing more than failed attempts to interpret the problem statement in a manner that is consistent with the conclusion that switching choices doubles the chances of winning. In my opinion, this is usually done because the claimant has solved the wrong problem and doesn't want to admit it or doesn't even realize that there really is more than one solution to the actual Monty Hall problem [Note 6]. For every argument that the actual Monty Hall problem should be interpreted as MHR1, there is an argument to the contrary. For example, “Surely Monty wouldn’t risk giving away twice as many prizes to obtain what is probably very little or no additional good will for the sponsors!” The most important thing to realize about these claims is that they consider only what the persons making such claims think is the right thing to do or something that they themselves would do rather than considering all possible strategies that may be employed by the game host, given the problem statement. This is a monumental error.

Myth: If the original game player of MHR1 is whisked away and replaced by a second game player who knows nothing about what happened previously and sees only two choices plus an unfavorable outcome, the second game player’s chances are 50/50 of obtaining a favorable outcome, so it doesn’t matter which choice he/she makes, therefore it doesn’t matter what choice the original (first) game player makes about switching.

Actually, for MHR1, it is only the average probability that is 50% for the second game player. The probability for each individual instance is actually either one-third or two-thirds, with equal likelihood for either one of these because the second game player doesn’t know which choice is the first game player’s original choice. In other words, if the second game player chooses the first game player’s original choice as their one and only choice, the probability of winning is one-third; if they choose the first game player’s switch choice instead, the probability of winning is two-thirds. On average, he/she will choose the first game player’s original choice half the time and the first game player’s switch choice the other half of the time. The quantity one-half times one-third plus the quantity one-half times two-thirds equals one-half or 50%.

A related issue: Suppose you are invited to make one of two choices and there is no offer to switch. Are your chances always 50/50? Not necessarily. Suppose that a prize is always placed in the position to the left and that a booby prize is always placed in the position to the right. Of course, you don’t know this, so you think that your chances are equal for either choice. If you always favor right over left when given a choice of any kind regarding left and right, you have a 100% chance of losing, not a 50% chance. Even if you had a strong tendency (say 75%) to select right over left, you would have a much worse chance than 50% of winning. Just because there are two choices does not necessarily mean that there is a 50% chance of making the correct choice. This example also illustrates why it’s so important to explicitly state that the favorable outcome in MHR1 is randomly placed, despite the fact the experiments show that this is difficult for humans to do unassisted.

Getting back to the situation with three choices and a second game player suddenly appearing: The second game player has the same odds as does the first game player for each individual instance, even if the problem is the actual Monty Hall problem at the time when the question is posed, but the average probability will always be one-half because they never know which choice is the original and which choice is the switch choice.

The average probability is also one-half for the first game player if he/she randomly decides whether or not to switch at each opportunity. This goes for all versions of the problem. For the actual Monty Hall problem, because the odds are unknown, a rational strategy for the one and only game player would be to flip a fair coin and decide based on that whether or not to switch. Even for MHR1, the game player’s chances of winning would increase from one-third to one-half on average (for many games). This is not the best strategy for winning in MHR1, but somewhat amusingly it does lead to an unexpected result that many consider to be unobtainable for MHR1. Of course, the answer to the question about whether or not there is an advantage for switching is still “yes” for MHR1.

Myth: Marilyn vos Savant "more explicity than previously" stated condition 2) in a followup to her original Parade Magazine article [Rodehouse]. Well, technically she did, but she was not explicit at all in the original problem statement, so this claim is extremely misleading because the term "more explicitly" implies that she was previously explicit which isn't the least bit true. If vos Savant had stated MHR1 properly, she would have undoubtedly faced less criticism. She left out all three required conditions for MHR1 in her original article. She described what I have been calling the actual Monty Hall problem, then provided a solution to MHR1.

Myth: The problem statements such as that by Marilyn vos Savant are not really defective; the problem is instead that people have a hard time understanding MHR1 [Rodehouse] (as I read his arguments around page 150). Yes, many people do have a problem understanding MHR1 as discussed above, but contrary to this myth, the problem statements are indeed defective in that they don't match the solution offered (or the other way around as I have been interpreting the situation). What is true however, is that many people recast real-world problems into solvable problems so that they can pontificate on them. This doesn't occur nearly as often in industry as it does in academia. Caveat student!

To quote an author of a textbook on the theory of probability many years ago, discussing another ambiguous problem: "The problem as stated in this form is not yet sufficiently precise to admit a unique solution." [Cramer] That's exactly what we have with most statements of the Monty Hall problem.

Myth: In the book [Rodehouse], the author describes Gardner's Three-Prisoner-Dilemma problem and notes the "great care" that Gardner took in stating that the warden was to flip a coin in the event that he could name either prisoner B or prisoner C, suggesting that Gardner fully understood the importance of doing this. The evidence says otherwise [Gardner, 1961], [Gardner, 1992], and [Gardner, 2001].

1) In later discussing the proffered solution to the three-prisoner dilemma, Gardner refers to a situation in which three cards, one red and two black, are dealt to three people. He does not mention anything about randomly picking one of the two black cards to turn over when there are two possibilities. 2) When Gardner discusses a three-shells game, he doesn't say anything about randomly choosing one of two empty shells when the game player selects the shell containing the pea. 3) When Gardner discusses Marilyn vos Savant's version of the actual Monty Hall problem, he doesn't say anything about the importance of randomly choosing one of two doors when the game player selects the door with the favorable outcome.

If Gardner fully understood the importance of condition 3) of MHR1, he should have mentioned its occurrence in all four of these cases, not just in one of them, and he should have discussed why it is important in all of them because it is not immediately obvious to most readers.

I strongly suspect that Gardner's reason for mentioning the flipping of a coin has to do with the fact that this detail was contained in versions of the problem that Gardner said had been "making the rounds" and that this appealed to his sense of symmetry, and not at all that he fully understood its importance in this and related problems. It is true that Gardner appreciated being unambiguous in general when stating problems, just not this particular detail. (Also, Gardner left out condition 2) of MHR1 for the three-prisoner problem, so this problem was still stated imprecisely enough to prevent a unique solution. That is, Gardner left open the question as to whether or not the warden had a choice about revealing anything to prisoner A.)

Myth: The eminent mathematician Paul Erdös was initially “wrong” when he said that the solution to the Monty Hall problem was that the odds were 50/50 for switching; he finally was persuaded by a simulation that he was “wrong.” In the biography of Paul Erdös by Paul Hoffman [Hoffman], the story of Erdös’s involvement with the Monty Hall problem is told as related by Andrew Vázsonyi. At first Vázsonyi claims that Erdös said, about the actual Monty Hall problem, that there should be no difference between switching and not switching choices. This answer actually is just as good a candidate as the answer given by Marilyn vos Savant who claimed that there is a difference. Both are possibilities; neither is true in general. If Erdös was “wrong,” so was vos Savant. In this biography, it is also stated that after being shown some simulations by Vázsonyi of MHR1 (here again, the wrong problem), Erdös allegedly admitted he was wrong. However, it is also stated that he was not actually satisfied with these results and later demanded the “Book proof” from his friend Ronald Graham who told him that the game host will always open a door revealing an unfavorable outcome, an explanation which Erdös purportedly accepted.

In his biography of Erdös [Schechter], Bruce Schechter tells a story very similar to Hoffman’s, but errs in adding his own declaration that “switching doors is by far the best course of action” for the actual Monty Hall problem, the same blunder that Steve Selvin and Marilyn vos Savant made, posing one problem then solving another. In other words, in Schechter’s view, Erdös was not only wrong but obviously wrong.

To be fair to Paul Erdös, the myth should be changed to: Paul Erdös solved a different problem from the one that was posed, just as did Marilyn vos Savant and Steve Selvin. When presented with simulations of MHR1, he was temporarily persuaded that he was wrong, but was still not fully convinced of it. When presented with addition of condition 2) of MHR1 to the actual Monty Hall problem, he agreed that it was better to switch for that particular problem.

Basically, Erdös was persuaded by others that the problem they were all supposed to be solving was a different problem (MHR1 minus conditions 1 and 3) than the one he was solving (MHR2) and accepted the solution for MHR1 as the solution for MHR1 minus conditions 1 and 3. He abandoned one solution to a problem different from the one posed to accept another solution that was for another problem different from the one posed, and also different from the one his friend Graham described.

If Erdös was “wrong,” then everyone in this entire story was “wrong,” and storytellers Hoffman and Schechter were “wrong” for declaring that vos Savant was correct.

In closing, consider a problem similar to the Monty Hall problem: You’re on an airplane, and you have a conversation with the woman sitting next to you. She says, “I have two children. One is a girl.” What are the odds that the other child is a girl? Some people who are familiar with this problem would like you to believe that the odds are one-third, based on probability theory (B-G plus G-B versus G-G). In certain other problems, this answer is defensible, but here it really isn’t. Do these people really expect the woman to perhaps continue, “And the other one is a girl?” Isn’t it virtually certain that she would continue, “And the other one is a boy” or perhaps, “And the other one identifies as an X” where X is anything other than “girl?”

When writing brain teasers and problems in textbooks, it is critically important to carefully write out all assumptions and avoid as much as possible misleading readers in the manner that I just did above.

Notes

[1] In this document, I have repeatedly reported that various authors have posed one problem, then solved another. I feel obligated to describe how Bruce Schechter, in his biography of Erdös, did this a second time, even though this may not be the best place to document it. After doing this with the actual Monty Hall problem, he introduced the concept of probabilities with respect to flipping of coins, then provided a solution for a specific case. He then went on to provide a different problem as if the solution he had already given was the solution to that second problem. As if this wasn't bad enough, he then went on to make an even bigger mistake, an unintentional bonus error.

"The probability of Guildenstern throwing 85 heads in a row is ½ x ½ x ½ x … 85 times or 1/2^85, which is about 1 in 4 x 10^25. That is another one of those numbers that is so large that it might as well be infinite. Even if he could toss his coin many trillions of times per second, Guildenstern could not reasonably expect such a run of heads before all the stars burned themselves to cinders." [Schechter, page 110].

The first problem discussed in the quote above has to do with the odds of a particular outcome in a series of outcomes, in particular 85 tosses of a coin per outcome. This Schechter stated correctly. What he didn't say is that the number of coin flips to perform corresponding to 4 x 10^25 outcomes is 85 x 4 x 10^25 or 3.4 x 10^27 coin flips. The second problem discussed in the quote, however, isn't about outcomes of 85 coin flips at a time; it's about calculating how long one continuous run of flips has to be to produce some reasonable probability of having a run of 85 heads in a row anywhere in the entire sequence. For example, in two outcomes of 85 coin flips, we have 170 coin flips. Within this run, there are 86 opportunities to have a run of 85 heads in a row, not just two. If this isn't clear, consider two outcomes of 2 coin flips each, four total coin flips. For outcomes, we could have HH, HT, TH, and TT where one in four outcomes is a run of two heads. Within the same four flips, however, we could have HHTX, THHT, or XTHH where X can be either H or T. There are three possibilities for two heads in a row, not just the two from the outcomes HH/TX and XT/HH. In addition, there are HHHT, THHH, and HHHH, each with multiple occurences of HH.

Using a simple approximation by Schilling [Schilling], the expected length of the longest run for 7.7 x 10^25 flips is close to 85. This is almost 45 times lower than the number 3.4 x 10^27 coin flips above.

The second mistake made by Schechter is the calculation of the length of time to produce the required number of flips. "Many trillions" of flips per second would be many times 10^12, say 320 x 10^12 = 3.2 x 10^14. This is three hundred and twenty trillion. The number of seconds in a year is approximately 3.2 x 10^7, so the number of coin flips in a year is about 10^22 at the rate of 3.2 x 10^14 flips per second. Dividing 7.7 x 10^25 flips by 10^22 flips per year results in 7,700 years. This is hardly the length of time that it would take for all the stars to have "burned themselves into cinders," or at least we should all hope so! Schechter's estimate is many orders of magnitude in error.

Lesson: Mistakes in calculations in probability can be off by huge amounts, so great care is needed in translating the problem into mathematics, solving the mathematical problem, and doing numerical calculations.

[2] Because so many people do not understand MHR1 even when conditions 1), 2), and 3) are explicitly stated, many others accuse those of us who complain (about authors posing one problem and solving another) of being too simple-minded to understand MHR1. This is an egregious error. Most of us who make this type of complaint understand the various forms of this problem far better than the vast majority of people who pontificate on it. If anyone thinks it's difficult to explain MHR1 to people, just try explaining what I'm saying here to someone who appeals to a supposed authority such as Martin Gardner or Marilyn vos Savant or to their college textbook on probability and statistics regarding the Monty Hall problem, or the three-shells problem, etc.

[3] Suppose there are a very large number of doors, behind only one of which is a favorable outcome. The game host opens all but two of them (one of them being the game player's original choice) before offering the game player the opportunity to switch to the other one. Argument: Surely the game player should switch, should they not? The chances of their original choice containing the favorable outcome is so very tiny. Alternative argument: Assume for the sake of this argument that there are as many players as doors and that all have chosen a door. At the point where you, for example, are asked if you want to switch, one other game player remains. Because you should both switch, the probability of obtaining the favorable outcome for each of you must necessarily be 50%, so there actually must be no advantage for either of you. Both of these arguments are invalid as shown above. Either conclusion may be correct, and it's possible that neither conclusion is correct, and that the game player has a disadvantage for switching.

[4] The actual answer to all versions of the Monty Hall problem is that if the game player has not selected the choice with the favorable outcome, then he/she should switch, otherwise not. Of course, this doesn't really help, and that's one point of the exercise.

Also, there is but one game to consider, so some would argue that probabilities don't apply. In this document, I have taken the approach in which we ask what would happen if the game was played an infinite number of times, taking note of the possibility of alternate behaviors, and I have avoided discussions of a more philosophical nature such as what we know and when we know it versus what actually is the case, etc. My reason for this is that there are more sophisticated problems in which this limited but practical approach is the only approach that one can use, so far as we know. There are many entities which don't have any existence, for example, other than in a statistical or probabilistic sense, so the question about what is actually the case in a particular instance doesn't apply.

[5] It's probably obvious in reading this document to see that I don't care at all for the MHR1 interpretation of the actual Monty Hall problem. The reason for this is that I view MHR1 as a simple-minded interpretation that is misleading about the way in which the real world works in many important situations.

[6] It seems to me that many people who advocate the MHR1 interpretation of the actual Monty Hall problem have assumed that the behavior of the host that is described in the problem is always the same every time the game is played, and in particular assume that the game host is always benevolent. Doesn't it ever occur to these people that there are other possible behaviors that could have been exhibited by the host if a second or third game had been played? In many important scientific and technological problems, it is important to consider all possible paths to a certain condition, not just the

one observed for a single trial. In my opinion, many of the invalid arguments that are given to support the MHR1 interpretation for the actual problem arise from a lack of realization that there are other possibilities that need to be considered whenever all three of the conditions for MHR1 are not explicitly stated.

[7] For corrections and/or comments, please email me at the address at the top of this document.

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