Green's Function Method for Creating Accurate Stereo Sound Images: Volumetric Sources

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02 July 2004, 28 August 2005

From a numerical standpoint, a point source is not necessarily a good type of source to use for solving the wave equation in three dimensions, nor is a point source necessarily a good physical source. Herein is developed a solution for simple volumetric sources for use in the Green's function method described in [1].

Consider the formal solution derived in [1] for the inhomogeneous wave equation, with homogeneous boundary and initial conditions:

$$\psi(\mathbf{r},t) = c^2 \sum_n{}' \left[\Phi_n(\mathbf{r}) \int_V u(\mathbf{r'})\ \Phi_n^*(\mathbf{r'}) \int_{t'=t_0}^{t+} s(t')\ \frac{\sin(\omega_n \tau)}{\omega_n}\ \theta(\tau)\ dt'\, dV' \right]$$

The product $u(\mathbf{r'})\ s(t')$ comprises the source term $\rho(\mathbf{r'}, t')$ for an audio signal; t_0 and $t+$ are the start and current times for the signal source, respectively, where we consider quiescent initial conditions; $t+ = t + \varepsilon$ and we will take the limit of $t+$ as $\varepsilon \to 0$ at the end of the calculation, once again following [2]. $\theta(t)$ is the Heaviside step function.

Once again the key to efficiently solving a number of important technological problems is that for certain cases it is possible to integrate, relatively easily and prior to computation, the integral:

$$\int_V u(\mathbf{r'})\ \Phi_n^*(\mathbf{r'})\ dV'$$

For example, consider a volumetric source extending from x_1 to x_2, y_1 to y_2, z_1 to z_2 inside a parallelepiped with sides a, b, and c in Cartesian coordinates and $u(\mathbf{r'}) = u =$ constant. The source term $\rho(\mathbf{r'},t) = u(\mathbf{r'})\, s(t')$ becomes (with $n \to nml$):

$$u(\mathbf{r'})\, s(t') = u\ [\theta(x' - x_1) - \theta(x' - x_2)]\ [\theta(y' - y_1) - \theta(y' - y_2)]\ [\theta(z' - z_1) - \theta(z' - z_2)]\ s(t')$$

$$\Phi_{nml}^*(\mathbf{r'}) = \sqrt{\frac{8}{abc}}\ \sin\left(\frac{n\pi x'}{a}\right) \sin\left(\frac{m\pi y'}{b}\right) \sin\left(\frac{l\pi z'}{c}\right)$$

The volume integral then becomes:

$$\int_V u\ [\theta(x' - x_1) - \theta(x' - x_2)]\ [\theta(y' - y_1) - \theta(y' - y_2)]\ [\theta(z' - z_1) - \theta(z' - z_2)]$$
$$\sqrt{\frac{8}{abc}}\ \sin\left(\frac{n\pi x'}{a}\right) \sin\left(\frac{m\pi y'}{b}\right) \sin\left(\frac{l\pi z'}{c}\right)\ dV'$$

$$= u \frac{\sqrt{8abc}}{nml} \left(\frac{1}{\pi^3}\right) \left[\cos\left(\frac{n\pi x_1}{a}\right) - \cos\left(\frac{n\pi x_2}{a}\right)\right]\left[\cos\left(\frac{m\pi y_1}{b}\right) - \cos\left(\frac{m\pi y_2}{b}\right)\right]\left[\cos\left(\frac{l\pi z_1}{c}\right) - \cos\left(\frac{l\pi z_2}{c}\right)\right]$$

With renormalization for the small volume, the solution now becomes:

$$\psi(\mathbf{r},t) = c^2 \frac{8u}{nml} \left(\frac{1}{\pi^3}\right) \left(\frac{1}{x_2 - x_1}\right) \left(\frac{1}{y_2 - y_1}\right) \left(\frac{1}{z_2 - z_1}\right)$$

$$\sum_{nml}{}' \left[\cos\left(\frac{n\pi x_1}{a}\right) - \cos\left(\frac{n\pi x_2}{a}\right)\right] \left[\cos\left(\frac{m\pi y_1}{b}\right) - \cos\left(\frac{m\pi y_2}{b}\right)\right]\left[\cos\left(\frac{l\pi z_1}{c}\right) - \cos\left(\frac{l\pi z_2}{c}\right)\right]$$

$$\left[\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right)\sin\left(\frac{l\pi z}{c}\right)\right] \int_{t'=-\infty}^{t} s(t') \frac{\sin(\omega_{nml}\tau)}{\omega_{nml}} dt'$$

similar to the expression in [1]. This resulting expression above can be used for computation. For a fixed reception point at (x,y,z), an impulse response function can be precomputed for the impulse signal $s(t') = \delta(t')$. Let

$$A_{nml} = c^2 \frac{8u}{nml} \left(\frac{1}{\pi^3}\right) \left(\frac{1}{x_2 - x_1}\right) \left(\frac{1}{y_2 - y_1}\right) \left(\frac{1}{z_2 - z_1}\right)$$

$$\left[\cos\left(\frac{n\pi x_1}{a}\right) - \cos\left(\frac{n\pi x_2}{a}\right)\right] \left[\cos\left(\frac{m\pi y_1}{b}\right) - \cos\left(\frac{m\pi y_2}{b}\right)\right]\left[\cos\left(\frac{l\pi z_1}{c}\right) - \cos\left(\frac{l\pi z_2}{c}\right)\right]$$

$$\left[\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right)\sin\left(\frac{l\pi z}{c}\right)\right]$$

which can now be computed for all n, m, l. Development and use of the impulse response function defined by

$$f(t) = \sum_{nml}{}' \frac{A_{nml}}{\omega_{nml}} \sin(\omega_{nml} t)$$

is identical from this point on to that in [1].

The expression for a point source can be derived from the expression for A_{nml} above by noting that

$$\lim_{x_2 \to x_1} \left[\left(\frac{1}{n}\right) \left(\frac{1}{\pi}\right) \left(\frac{1}{x_2 - x_1}\right)\left[\cos\left(\frac{n\pi x_1}{a}\right) - \cos\left(\frac{n\pi x_2}{a}\right)\right]\right] = - \left(\frac{1}{n\pi}\right)\frac{d}{dx_1}\left[\cos\left(\frac{n\pi x_1}{a}\right)\right]$$

$$= \frac{1}{a} \sin\left(\frac{n\pi x_1}{a}\right)$$

which is the form of the comparable terms obtained for the point source in [1].

References

[1] David R. Clark. Green's function method for creating accurate stereo sound images. EXE Consulting, P.O. Box 450998, Garland, Texas 75045-0998, July 2004.

[2] Gabriel Barton. *Elements of Green's Functions and Propagation: Potentials, Diffusion, and Waves.* Oxford, 1989. Reprinted 1991.