Green's Function Method for Creating Accurate Stereo Sound Images: Simple Three-Dimensional Example Rectangular Parallelepiped

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Herein is presented an example which shows the Green's function method described in [1] applied to a simple, three-dimensional, pedagogical example. This example could be interpreted as the striking of a small rectangular volume or rectangular area inside a rectangular parallelepiped and picking up the response at a point location elsewhere in the parallelepiped. This approach has been used by this author to calculate impule responses for rooms.

We start from the solution for the three-dimensional (3-D) wave equation derived in [1] and follow the same format as for the one-dimensional example [3] except that we will also adopt the volumetric source as in [4]. The 3-D solution for a volumetric source is (noting that *c* is the speed of sound in the medium while c is the height of the parallelepiped):

$$\psi(\mathbf{r},t) = c^2 \frac{8u}{nml}\left(\frac{1}{\pi^3}\right)\left(\frac{1}{x_2-x_1}\right)\left(\frac{1}{y_2-y_1}\right)\left(\frac{1}{z_2-z_1}\right)$$

$$\sum_{nml}{}' \left[\cos\left(\frac{n\pi x_1}{a}\right)-\cos\left(\frac{n\pi x_2}{a}\right)\right]\left[\cos\left(\frac{m\pi y_1}{b}\right)-\cos\left(\frac{m\pi y_2}{b}\right)\right]\left[\cos\left(\frac{l\pi z_1}{c}\right)-\cos\left(\frac{l\pi z_2}{c}\right)\right]$$

$$\left[\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right)\sin\left(\frac{l\pi z}{c}\right)\right]\int_{t'=-\infty}^{t} s(t')\ \frac{\sin(\omega_{nml}\tau)}{\omega_{nml}}\ dt'$$

For a fixed reception point at (x,y,z), an impulse response function can be precomputed for a source impulse signal δ (t').

Let

$$A_{nml} = c^2 \frac{4u}{nml}\left(\frac{1}{\pi^2}\right)\left(\frac{1}{x_2-x_1}\right)\left(\frac{1}{y_2-y_1}\right)\left(\frac{1}{z_2-z_1}\right)$$

$$\left[\cos\left(\frac{n\pi x_1}{a}\right)-\cos\left(\frac{n\pi x_2}{a}\right)\right]\left[\cos\left(\frac{m\pi y_1}{b}\right)-\cos\left(\frac{m\pi y_2}{b}\right)\right]\left[\cos\left(\frac{l\pi z_1}{c}\right)-\cos\left(\frac{l\pi z_2}{c}\right)\right]$$

$$\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right)\sin\left(\frac{l\pi z}{c}\right)$$

which can be computed for all n,m,l if the input data are known, and let

$$f(t) = \int_{t'=-\infty}^{t} \sum_{nml}{}' \left[A_{nml} \frac{\sin(\omega_{nml}\tau)}{\omega_{nml}} \right] \delta(t')\, dt'$$

$$= \sum_{nml}{}' \frac{A_{nml}}{\omega_{nml}} \sin(\omega_{nml} t)$$

Now we have the impulse response function for this problem, namely the discrete sin transform of the quantity $\frac{A_{nml}}{\omega_{nml}}$. In any real computation, the number of frequencies is finite and f(t) is periodic.

As described in [1], but adapted to this 3-D example, to compute the resulting sound received at point (x, y, z) over time from a volumetric source extending from x_1 to x_2, y_1 to y_2, and z_1 to z_2 one could compute the convolution:

$$f(t) * s(t) \equiv \int_{-\infty}^{\infty} f(t')\, s(t-t')\, dt' = \int_{-\infty}^{\infty} f(t-t')\, s(t')\, dt' = \int_{-\infty}^{\infty} f(\tau)\, s(t')\, dt' = \psi(x, y, z; t)$$

In order to compute $\frac{A_{nml}}{\omega_{nml}}$ it is first necessary to choose a cutoff frequency so that n, m, and l are finite.

$$\omega_{nml} = \sqrt{\left(\frac{n\pi c}{a}\right)^2 + \left(\frac{m\pi c}{b}\right)^2 + \left(\frac{l\pi c}{c}\right)^2} \quad \text{resulting in} \quad f_{nml} = \frac{\omega_{nml}}{2\pi} = \frac{c}{2} \sqrt{\left(\frac{n}{a}\right)^2 + \left(\frac{m}{b}\right)^2 + \left(\frac{l}{c}\right)^2} \ .$$

The fundamental frequency is given by:

$$f_{111} = \frac{c}{2} \sqrt{\left(\frac{1}{a}\right)^2 + \left(\frac{1}{b}\right)^2 + \left(\frac{1}{c}\right)^2}$$

Letting $R_1 = \frac{b}{a}$ and $R_2 = \frac{c}{a}$:

$$f_{111} = \frac{c\sqrt{R_1^2 R_2^2 + R_2^2 + R_1^2}}{2aR_1R_2}$$

But we also have:

$$f_{n11} = \frac{c\sqrt{n^2 R_1^2 R_2^2 + R_2^2 + R_1^2}}{2aR_1R_2} \quad , \quad f_{1m1} = \frac{c\sqrt{R_1^2 R_2^2 + m^2 R_2^2 + R_1^2}}{2aR_1R_2} \quad \text{and} \quad f_{111} = \frac{c\sqrt{R_1^2 R_2^2 + R_2^2 + l^2 R_1^2}}{2aR_1R_2}$$

Choosing $f_{111} = 110\,\text{Hz}$, the maximum value of n can be specified by considering:

$$\frac{f_{n11}}{f_{111}} = \frac{\sqrt{n^2 R_1^2 R_2^2 + R_2^2 + R_1^2}}{\sqrt{R_1^2 R_2^2 + R_2^2 + R_1^2}} = \frac{f_{max}}{110}$$

Solving for n:

$$n_{max} = \left(\frac{1}{R_1 R_2}\right) \sqrt{\left(\frac{f_{max}}{110}\right)^2 \left(R_1^2 R_2^2 + R_2^2 + R_1^2\right) - R_2^2 - R_1^2}$$

For any given m, l :

$$n_{max} = \left(\frac{1}{R_1 R_2}\right) \sqrt{\left(\frac{f_{max}}{110}\right)^2 \left(R_1^2 R_2^2 + R_2^2 + R_1^2\right) - m^2 R_2^2 - l^2 R_1^2}$$

Solving the corresponding expression for m_{max} gives:

$$m_{max} = \left(\frac{1}{R_2}\right) \sqrt{\left(\frac{f_{max}}{110}\right)^2 \left(R_1^2 R_2^2 + R_2^2 + R_1^2\right) - R_1^2 R_2^2 - R_1^2}$$

and for any particular n, l :

$$m_{max} = \left(\frac{1}{R_2}\right) \sqrt{\left(\frac{f_{max}}{110}\right)^2 \left(R_1^2 R_2^2 + R_2^2 + R_1^2\right) - n^2 R_1^2 R_2^2 - l^2 R_1^2}$$

Similarly, for l_{max} :

$$l_{max} = \left(\frac{1}{R_1}\right) \sqrt{\left(\frac{f_{max}}{110}\right)^2 \left(R_1^2 R_2^2 + R_2^2 + R_1^2\right) - R_1^2 R_2^2 - R_2^2}$$

and for any given n, m :

$$l_{max} = \left(\frac{1}{R_1}\right) \sqrt{\left(\frac{f_{max}}{110}\right)^2 \left(R_1^2 R_2^2 + R_2^2 + R_1^2\right) - n^2 R_1^2 R_2^2 - m^2 R_2^2}$$

The coefficients are now:

$$\frac{A_{nml}}{\omega_{nml}} = \frac{4uc}{nml} \left(\frac{1}{\pi^3}\right) \left(\frac{1}{\sqrt{\frac{n^2}{a^2} + \frac{m^2}{b^2} + \frac{l^2}{c^2}}}\right) \left(\frac{1}{x_2 - x_1}\right) \left(\frac{1}{y_2 - y_1}\right) \left(\frac{1}{z_2 - z_1}\right)$$

$$\left[\cos\left(\frac{n\pi x_1}{a}\right) - \cos\left(\frac{n\pi x_2}{a}\right)\right] \left[\cos\left(\frac{m\pi y_1}{b}\right) - \cos\left(\frac{m\pi y_2}{b}\right)\right] \left[\cos\left(\frac{l\pi z_1}{c}\right) - \cos\left(\frac{l\pi z_2}{c}\right)\right]$$

$$\sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{b}\right) \sin\left(\frac{l\pi z}{c}\right)$$

Normalizing this expression for n = m = l = 1, and once again utilizing R_1 and R_2, the ratios of b to a and c to a:

$$\frac{A_{nml}}{\omega_{nml}} = \left(\frac{\sqrt{R_1^2 R_2^2 + R_2^2 + R_1^2}}{nml \sqrt{R_1^2 R_2^2 n^2 + m^2 R_2^2 + l^2 R_1^2}} \right)$$

$$\left[\cos\left(\frac{n\pi x_1}{a}\right) - \cos\left(\frac{n\pi x_2}{a}\right)\right] \left[\cos\left(\frac{m\pi y_1}{b}\right) - \cos\left(\frac{m\pi y_2}{b}\right)\right] \left[\cos\left(\frac{l\pi z_1}{c}\right) - \cos\left(\frac{l\pi z_2}{c}\right)\right]$$

$$\sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{b}\right) \sin\left(\frac{l\pi z}{c}\right)$$

If we now specify the area determined by the normalized dimensions

$$\left(\frac{x_1}{a}\right), \left(\frac{x_2}{a}\right), \left(\frac{y_1}{b}\right), \left(\frac{y_2}{b}\right), \left(\frac{z_1}{c}\right), \text{ and } \left(\frac{z_2}{c}\right)$$

and if also we specify the normalized pickup locations $\left(\frac{x}{a}\right)$, $\left(\frac{y}{b}\right)$, and $\left(\frac{z}{c}\right)$ and the ratios R_1 and R_2 (of b to a and c to a) we can determine the coefficients $\frac{A_{nml}}{\omega_{nml}}$ and, by means of the sin transform, solve the 3-D wave equation for a rectangular parallelepiped.

References and Notes

[1] David R. Clark. Green's function method for creating accurate stereo sound images. EXE Consulting, P.O. Box 450998, Garland, Texas 75045-0998, July 2004.

[2] Gabriel Barton. *Elements of Green's Functions and Propagation: Potentials, Diffusion, and Waves.* Oxford, 1989. Reprinted 1991.

[3] David R. Clark. Green's function method for creating accurate stereo sound images: Simple one-dimensional example. EXE Consulting, P.O. Box 450998, Garland, Texas 75045-0998, July-August 2004.

[4] David R. Clark. Green's function method for creating accurate stereo sound images: Volumetric Sources. EXE Consulting, P.O. Box 450998, Garland, Texas 75045-0998, July 2004.

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