# Green's Function Method for Creating Accurate Stereo Sound Images

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A difficult problem in audio engineering is accurate simulation of concert halls and auditoriums for the production of realistic reverberation, echo, and stereo separation for audio sources. Finite difference and finite element approaches to this problem require large resources. Herein is developed an efficient method for creating accurate stereo images for simulating concert halls, auditoriums and other audio environments. The formal solution to the three-dimensional (3-D) wave equation is rewritten in a form more amenable to computation.

Following [1], we start from the formal solution for the inhomogeneous wave equation, with homogeneous boundary and initial conditions:

$$\psi(\mathbf{r},t) = \int_V \int_{t'=t_0}^{t+} G(\mathbf{r},t \mid \mathbf{r'},t')\ \rho(\mathbf{r'},t')\, dt'\, dV'$$

where $G(\mathbf{r},t \mid \mathbf{r'},t')$ is the Green's function for the wave operator $\frac{1}{c^2}\left(\frac{\partial^2}{\partial t^2} - \nabla^2\right)$ and for the given boundary and initial conditions; $\rho(\mathbf{r'},t')$ is the source term, and $t_0$ and $t+$ are the start and current times for the signal source, respectively, where we consider signals only of the type: $\rho(\mathbf{r'},0) = \rho_t(\mathbf{r'},0) = 0$ (*i.e.* quiescent initial conditions). $t+ = t + \varepsilon$ and we will take the limit of $t+$ as $\varepsilon \to 0$ at the end of the calculation, again following [1].

Assume that an audio source $\rho(\mathbf{r'},t')$ can be represented as a product of spatial and temporal functions:

$$\rho(\mathbf{r'}, t') = u(\mathbf{r'})\ s(t') \quad \forall \quad \mathbf{r'}, t'$$

In terms of the normalized eigenfunctions $\Phi_n(\mathbf{r})$ for the Laplace operator $-\nabla^2$, for the case of not purely Neumann boundary conditions (no term for n = 0) and homogeneous Cauchy initial conditions, $G(\mathbf{r},t \mid \mathbf{r'},t')$ can be expressed as:

$$G(\mathbf{r},t \mid \mathbf{r'},t') = c^2 \sum_n{}' \ \Phi_n^*(\mathbf{r'})\ \Phi_n(\mathbf{r})\ \frac{\sin(\omega_n \tau)}{\omega_n}\ \theta(\tau)$$

where $\tau = t - t'$ and $\theta(t)$ is the Heaviside step function:

$$\theta(t) = \begin{cases} 0 & \text{for } t<0 \\ 1 & \text{for } t>0 \end{cases} \quad \text{(weak definition).}$$

The solution is then:

$$\psi(\mathbf{r},t) = c^2 \sum_n{}' \left[ \Phi_n(\mathbf{r}) \int_V u(\mathbf{r'})\ \Phi_n^*(\mathbf{r'}) \int_{t'=t_0}^{t+} s(t')\ \frac{\sin(\omega_n \tau)}{\omega_n}\ \theta(\tau)\, dt'\, dV' \right]$$

The key to efficiently solving a number of important technological problems is that for certain cases it is possible to integrate, relatively easily and prior to computation, the integral:

$$\int_V u(\mathbf{r'})\ \Phi_n^*(\mathbf{r'})\ dV'$$

For example, consider a point source at (x", y", z") in a rectangular parallelepiped with sides a, b, and c in Cartesian coordinates and $u(\mathbf{r'}) = u$ = constant (with n → nml):

$$\rho(\mathbf{r'}, t') = u(\mathbf{r'})\ s(t') = u\ \delta(x'' - x')\ \delta(y'' - y')\ \delta(z'' - z')\ s(t')$$

$$\Phi_{nml}^*(\mathbf{r'}) = \sqrt{\frac{8}{abc}}\ \sin\left(\frac{n\pi x'}{a}\right)\sin\left(\frac{m\pi y'}{b}\right)\sin\left(\frac{l\pi z'}{c}\right)$$

The volume integral then becomes:

$$\int_V u\ \delta(x'' - x')\ \delta(y'' - y')\ \delta(z'' - z')\sqrt{\frac{8}{abc}}\ \sin\left(\frac{n\pi x'}{a}\right)\sin\left(\frac{m\pi y'}{b}\right)\sin\left(\frac{l\pi z'}{c}\right) dV'$$

$$= u\sqrt{\frac{8}{abc}}\ \sin\left(\frac{n\pi x''}{a}\right)\sin\left(\frac{m\pi y''}{b}\right)\sin\left(\frac{l\pi z''}{c}\right)$$

The solution now becomes:

$$\psi(\mathbf{r},t) = c^2\ \frac{8u}{abc}\sum_{nml}{}'\ \Bigg[\sin\left(\frac{n\pi x''}{a}\right)\sin\left(\frac{m\pi y''}{b}\right)\sin\left(\frac{l\pi z''}{c}\right)\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right)\sin\left(\frac{l\pi z}{c}\right)$$
$$\int_{t'=-\infty}^{t} s(t')\ \frac{\sin(\omega_{nml}\tau)}{\omega_{nml}}\ dt'\Bigg]$$

where we have replaced the limits of the time integral with $t_0 \to -\infty$ and $t+ \to t$ because s(t' ) will always be zero prior to time zero anyway, and t will always be greater than t'. We have also dropped $\theta(\tau)$ because t will always be greater than t' .

This resulting expression above can be used for computation. For a fixed reception point at (x,y,z), an impulse response function can be precomputed for the impulse signal $\delta$ (t' ).

Let

$$A_{nml} = c^2\ \frac{8u}{abc}\ \sin\left(\frac{n\pi x''}{a}\right)\sin\left(\frac{m\pi y''}{b}\right)\sin\left(\frac{l\pi z''}{c}\right)\sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right)\sin\left(\frac{l\pi z}{c}\right)$$

which can now be computed for all n, m, l and let

$$f(t) = \int_{t'=-\infty}^{t}\sum_{nml}{}'\left[A_{nml}\ \frac{\sin(\omega_{nml}\tau)}{\omega_{nml}}\right]\delta(t')\ dt'$$

$$= \sum_{nml}{}' \frac{A_{nml}}{\omega_{nml}} \sin(\omega_{nml} t)$$

Now we have the impulse response function for this problem, namely the discrete sin transform of the quantity $\frac{A_{nml}}{\omega_{nml}}$ .

To compute the resulting sound received at point (x,y,z) over time from a source at (x", y", z"), we can compute the convolution:

$$f(t) * s(t) \equiv \int_{-\infty}^{\infty} f(t') \, s(t - t') \, dt' = \int_{-\infty}^{\infty} f(t - t') \, s(t') \, dt' = \int_{-\infty}^{\infty} f(\tau) \, s(t') \, dt'$$

The only difference between the these integrals and the solution $\psi$(x,y,z;t) is the upper limit of integration. Therefore, let us consider the convolution for a particular time t, given the conditions we will place on f(t) and s(t' ). First, we have already specified that s(t' ) is zero prior to time zero. This is also true for f(t) for all cases that we will consider. That is, we will consider only "retarded" solutions, namely only those solutions wherein effects are propagated into the future. Therefore, not only is the integrand always zero for $t' < 0$, it is also always zero for $t' > t$. Due to the fact that c is finite, it is also zero for $t' = t$ as long as the reception point and the location of the source are separated in space. In effect, the upper limit of the integrand may be then taken as t rather than infinity at any particular time t. Therefore, for the cases under consideration, at any particular time t:

$$\psi(x,y,z;t) = f(t) * s(t)$$

If this is done for two locations representing the left and right ears in an auditorium, for example, a stereo image representing the solution to the wave equation for the two reception locations driven by a single source location can be formed:

$$f_L(t) * s(t) \cup f_R(t) * s(t) = s(t) * [f_L(t) \cup f_R(t)]$$

For multiple sources, a "mix" can be specified mathematically by:

$$\text{Mix} = \sum_k s_k(t) * [f_{k,L}(t) \cup f_{k,R}(t)]$$

What is needed is a set of coefficients (impulse response functions) $f_{k,L}(t)$ and $f_{k,R}(t)$ and separately recorded channels for each source position. The functions $f_{k,L}(t)$ and $f_{k,R}(t)$ can be (and have been) used repeatedly to simulate a particular listening environment to produce a very high quality stereo image with very accurate echo, reverberation, and stereo separation in accordance with the solution to the three- dimensional wave equation, subject to the boundary and initial conditions of the listening environment, for a wide variety of audio sources (recordings) in that same environment. As a practical note for a certain technological problem, this approach produces "out of the head" stereo audio in headphones, simulating binaural recordings. This is experienced as headphones sounding like external speakers to the listener.

Extensions to the discussion above include solutions for line, membrane, and volumetric sources as well as for cylindrical and spherical environments. Absorption can be comprehended either by using complex- valued eigenfunctions or, more simply, by multiplying the impulse response function by an exponentially decaying function in time. The approach described above can also be used to simulate instruments; indeed a concert hall is a 3- D instrument. The application to string and membrane instruments has not been discussed here but involves 1-D and 2-D solutions which result from trivial changes to the above solutions.

## Reference

[1] Gabriel Barton. *Elements of Green's Functions and Propagation: Potentials, Diffusion, and Waves.* Oxford, 1989. Reprinted 1991.