Green's Function Method for Creating Accurate Stereo Sound Images: Simple Two-Dimensional Example Circular Membrane

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Herein is presented an example which demonstrates the Green's function method described in [1] applied to a simple, two-dimensional, pedagogical example. This example could be interpreted as the striking of a point location on a circularly shaped membrane, for example a drumhead, and picking up the response at a point location elsewhere on the membrane.

We start from the solution for the three-dimensional (3-D) wave equation derived in [1] and [2] for which the background is [3]. In [2], the solution was:

$$\psi(\mathbf{r},t) = c^2 \frac{4u\,r''}{Z\,\pi\,a^2} \sum_{nml}{}' \left[\frac{1}{J^2_{n+1}(\beta_{nm}\,a)} J_n(\beta_{nm}\,r'')\cos(n\,\phi'')\sin\left(\frac{l\pi z''}{Z}\right) J_n(\beta_{nm}\,r)\cos(n\,\phi)\sin\left(\frac{l\pi z}{Z}\right) \int_{t'=-\infty}^{t} s(t')\,\frac{\sin(\omega_{nml}\,\tau)}{\omega_{nml}}\,dt' \right]$$

where the modal frequencies are:

$$\omega_{nml} = \sqrt{(\beta_{nm}\,a)^2\left(\frac{c}{a}\right)^2 + \left(\frac{l\pi z}{Z}\right)^2}$$

For 2-D, these become:

$$\psi(\mathbf{r},t) = c^2 \frac{2u\,r''}{\pi\,a^2} \sum_{nm}{}' \left[\frac{1}{J^2_{n+1}(\beta_{nm}\,a)} J_n(\beta_{nm}\,r'')\cos(n\,\phi'') J_n(\beta_{nm}\,r)\cos(n\,\phi) \int_{t'=-\infty}^{t} s(t')\,\frac{\sin(\omega_{nm}\,\tau)}{\omega_{nm}}\,dt' \right]$$

and the modal frequencies are:

$$\omega_{nm} = (\beta_{nm}\,a)\left(\frac{c}{a}\right)$$

Let

$$A_{nm} = c^2 \frac{2u\, r''}{\pi a^2} \frac{1}{J_{n+1}^2(\beta_{nm}\, a)} J_n(\beta_{nm}\, r'') \cos(n\phi'') J_n(\beta_{nm}\, r) \cos(n\phi)$$

which can now be computed for all n and m. The impulse response function is expressed by:

$$f(t) = \sum_{nm}{}' \frac{A_{nm}}{\omega_{nm}} \sin(\omega_{nm} t)$$

Now we need to determine the limits for n and m, given a maximum frequency. In this case of the circular membrane, the fundamental frequency is given by:

$$\omega_{01} = (\beta_{01}\, a)\left(\frac{c}{a}\right)$$

where $(\beta_{01}\, a) \approx 2.405$ is the first zero of the zeroth order Bessel function, c is the speed of sound in the membrane, and a is the radius of the circular membrane. Subsequent zeros for the zeroth order Bessel function are approximately $(\beta_{0m}\, a) \approx 2.405 + m\pi$, so:

$$\omega_{0m} \approx (2.405 + m\pi)\left(\frac{c}{a}\right)$$

Because the index m becomes largest for the zeroth order Bessel function, the largest value of m (m_{max}) can be obtained from the approximate expression for f_{max}:

$$\omega_{max} = 2\pi f_{max} \approx (2.405 + m_{max}\pi)\left(\frac{c}{a}\right)$$

As the order of the Bessel function increases, fewer and fewer frequencies contribute to the spectrum until none do. Because the zeros are interlaced between successive orders and because the spacing of the zeros is approximately π (*i.e.* not diminishing significantly), the maximum order of the Bessel functions which contribute to the spectrum (n_{max}) is not greater than the number of zeros which need to be considered from the zeroth order Bessel function; therefore, $n_{max} \le m_{max}$. Because the expression for m_{max} is only an approximate one, we include the possibility of a relationship of equality here.

Summarizing the recent discussion:

$$n_{max} \leqslant m_{max} \approx \frac{2 f_{max}\, a}{c} - \frac{2.405}{\pi}$$

Returning to the expression for the fundamental frequency:

$$\omega_{01} = (\beta_{01}\, a)\left(\frac{c}{a}\right) = 2\pi f_{01}$$

Once a fundamental frequency is chosen, we can determine the value of $\left(\frac{a}{c}\right)$:

$$\frac{a}{c} = \frac{(\beta_{01}\ a)}{2\pi f_{01}} \approx \frac{0.3828}{f_{01}}$$

Now we can use this expression to determine n_{max} and m_{max} even more directly:

$$n_{max} \leqslant m_{max} \approx \frac{f_{max}(\beta_{01}\ a)}{\pi f_{01}} - \frac{(\beta_{01}\ a)}{\pi} = \frac{(\beta_{01}\ a)\ (f_{max} - f_{01})}{\pi f_{01}} \approx 2.405\ \frac{(f_{max} - f_{01})}{\pi f_{01}}$$

This result can be derived more simply, but doing it in this roundabout way allows discussion of some important relations. Now again thanks to the interlacing and more or less even spacing of zeros, n_{max} for any given m is:

$$n_{max} \approx 2.405\ \frac{(f_{max} - f_{01})}{\pi f_{01}} - m$$

A similar expression holds for m_{max} for any given n. These results will get us fairly close to all of the frequencies that we need to consider.

In order to obtain more uniform results for choices of c, a, r, and r'', we can normalize the expression for A_{nm}:

$$A_{nm} = \frac{1}{J_{n+1}^2(\beta_{nm}\ a)}\ J_n(\beta_{nm}\ r'')\ \cos(n\,\phi'')\ J_n(\beta_{nm}\ r)\ \cos(n\,\phi)$$

Making substitutions so that we can express dimensions r and r'' as fractions of the radius, and allowing c to take on the value unity:

$$A_{nm} = \frac{1}{J_{n+1}^2(\beta_{nm}\ a)}\ J_n\left(\beta_{nm}\left(\frac{r''}{a}\right)\ \frac{(\beta_{01}\ a)}{2\pi f_{01}}\right)\ \cos(n\,\phi'')\ J_n\left(\beta_{nm}\left(\frac{r}{a}\right)\ \frac{(\beta_{01}\ a)}{2\pi f_{01}}\right)\ \cos(n\,\phi)$$

Approximately:

$$A_{nm} \approx \frac{1}{J_{n+1}^2(\beta_{nm}\ a)}\ J_n\left(\beta_{nm}\left(\frac{r''}{a}\right)\ \frac{0.3828}{f_{01}}\right)\ \cos(n\,\phi'')\ J_n\left(\beta_{nm}\left(\frac{r}{a}\right)\ \frac{0.3828}{f_{01}}\right)\ \cos(n\,\phi)$$

Because

$$\omega_{nm} = (\beta_{nm}\ a)\left(\frac{c}{a}\right) = (\beta_{nm}\ a)\left(\frac{c}{a}\right)\ \frac{\omega_{01}}{\omega_{01}} = \frac{(\beta_{nm}\ a)\left(\frac{c}{a}\right)}{(\beta_{01}\ a)\left(\frac{c}{a}\right)}\ \omega_{01} = \frac{(\beta_{nm}\ a)}{(\beta_{01}\ a)}\ (2\pi f_{01})$$

the approximate normalized expression for the coefficients $\frac{A_{nm}}{\omega_{nm}}$ is:

$$\frac{A_{nm}}{\omega_{nm}} \approx \left(\frac{(\beta_{01}\ a)}{(\beta_{nm}\ a)}\right)\ \frac{1}{J_{n+1}^2(\beta_{nm}\ a)}\ J_n\left(\beta_{nm}\left(\frac{r''}{a}\right)\ \frac{0.3828}{f_{01}}\right)\ \cos(n\,\phi'')\ J_n\left(\beta_{nm}\left(\frac{r}{a}\right)\ \frac{0.3828}{f_{01}}\right)\ \cos(n\,\phi)$$

The first order of business in calculating the coefficients A_{nm} is to determine the value of β_{nm}. This is done by determining the argument of the m^{th} zero of the n^{th} order Bessel function, from which β_{nm} can be calculated. For example:

$$(\beta_{01}\, a) \approx 2.405 \approx \beta_{01} \frac{0.3828}{f_{01}}$$

The algorithm to calculate these coefficients involves determining the values of zeros of Bessel functions, calculating β_{nm}, then using these values of β_{nm} in the arguments of Bessel functions to determine the coefficients A_{nm}. Once this is done, then the sin transform is taken to determine the impulse response function which can then be convolved with a source function to produce the solution. This is the same strategy that was used in previous notes, but made somewhat more complicated by use of Bessel functions.

References and Notes

[1] David R. Clark. Green's function method for creating accurate stereo sound images. EXE Consulting, P.O. Box 450998, Garland, Texas 75045-0998, July 2004.

[2] David R. Clark. Green's function method for creating accurate stereo sound images: cylindrical volume. EXE Consulting, P.O. Box 450998, Garland, Texas 75045-0998, July 2004.

[3] Gabriel Barton. *Elements of Green's Functions and Propagation: Potentials, Diffusion, and Waves.* Oxford, 1989. Reprinted 1991.