Green's Function Method for Creating Accurate Stereo Sound Images: Simple One-Dimensional Example

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Herein is presented an example which shows the Green's function method described in [1] applied to a simple, one-dimensional, pedagogical example. An interpretation is suggested for this example so that the formalism can be related to something more concrete.

We start from the solution for the three-dimensional (3-D) wave equation derived in [1]:

$$\psi(\mathbf{r},t) = c^2 \frac{8u}{abc} \sum_{nml}{}' \left[\sin\left(\frac{n\pi x''}{a}\right) \sin\left(\frac{m\pi y''}{b}\right) \sin\left(\frac{l\pi z''}{c}\right) \sin\left(\frac{n\pi x}{a}\right) \sin\left(\frac{m\pi y}{b}\right) \sin\left(\frac{l\pi z}{c}\right) \int_{t'=-\infty}^{t} s(t') \frac{\sin(\omega_{nml}\tau)}{\omega_{nml}} \, dt' \right]$$

and modify it for the case of 1-D:

$$\psi(x,t) = c^2 \frac{2u}{a} \sum_{n}{}' \left[\sin\left(\frac{n\pi x''}{a}\right) \sin\left(\frac{n\pi x}{a}\right) \int_{t'=-\infty}^{t} s(t') \frac{\sin(\omega_n \tau)}{\omega_n} \, dt' \right]$$

For a fixed reception point at (x), an impulse response function can be precomputed for a source impulse signal $\delta(t')$.

Let

$$A_n = c^2 \frac{2u}{a} \sin\left(\frac{n\pi x''}{a}\right) \sin\left(\frac{n\pi x}{a}\right)$$

which can now be computed for all n, and let

$$f(t) = \int_{t'=-\infty}^{t} \sum_{n}{}' \left[A_n \frac{\sin(\omega_n \tau)}{\omega_n} \right] \delta(t') \, dt'$$

$$= \sum_{n}{}' \frac{A_n}{\omega_n} \sin(\omega_n t)$$

Now we have the impulse response function for this problem, namely the discrete sin transform of the quantity $\frac{A_n}{\omega_n}$. In any real computation, the number of frequencies is finite and f(t) is periodic.

As described in [1], but adapted to this 1-D example, to compute the resulting sound received at point (x) over time from a source at (x''), one could compute the convolution:

$$f(t) * s(t) \equiv \int_{-\infty}^{\infty} f(t')\, s(t - t')\, dt' = \int_{-\infty}^{\infty} f(t - t')\, s(t')\, dt' = \int_{-\infty}^{\infty} f(\tau)\, s(t')\, dt' = \psi(x, t)$$

However, for the case of $s(t') = \delta(t')$:

$$\psi(x, t) = f(t) \quad \text{(x is now a fixed point).}$$

That is, the solution for the case of an impulse at position (x'') on a string, at another location (x) on the string, is the impulse response function itself, as is expected. We could interpret the impulse at position (x'') as the very abrupt strumming of a guitar string or the very abrupt striking of a keyboard wire. The solution at position (x) could be interpreted as the response at the location of the pickup of that guitar string or keyboard wire, therefore as the sound of the instrument under those conditions, prior to amplification, reverb, or other processing. If we do this for several different lengths of the string or wire, we can constuct a complete instrument such as a guitar or keyboard. An accurate model of a guitar string or keyboard wire would require a slightly more complex model; nevertheless, instruments can certainly be constructed from the procedure described here.

We proceed in three steps: First compute the values of $\frac{A_n}{\omega_n}$; second compute the discrete sin transform of $\frac{A_n}{\omega_n}$. Due to the fact we don't need to perform the convolution for this case, we will then have a waveform that represents a note that can be played by a 1-D instrument. The third step is to multiply the solution $\psi(x, t)$, here $f(t)$, by a function that decays in time. Due to the periodicity of $f(t)$, the characteristic decay time needs to be short enough so that $f(t)$ decays to an inaudible level before any periodic effects become obvious. To accomplish this, we can use the function $e^{-\alpha t}$ where α is chosen to cut the note off in time [3].

In order to compute $\frac{A_n}{\omega_n}$ it is first necessary to choose a cutoff frequency so that n is finite.

$$\omega_n = \frac{n \pi c}{a} \quad \text{resulting in} \quad f_n = \frac{\omega_n}{2\pi} = \frac{n c}{2a} .$$

The fundamental frequency is given by:

$$f_1 = \frac{c}{2a}$$

Choosing $f_1 = 440\text{Hz}$:

$$n = f_n \frac{2a}{c} = \frac{f_n}{440}$$

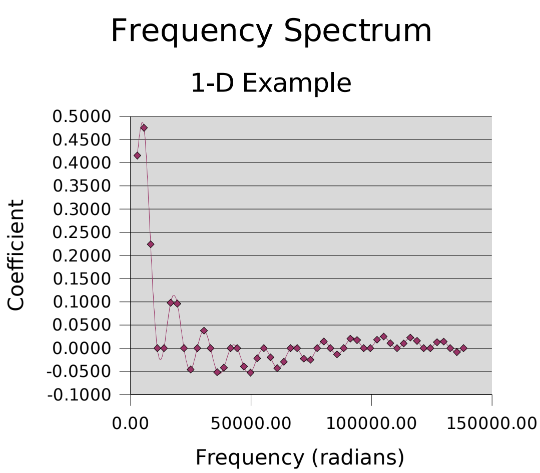
For a maximum frequency of 22,050 Hz, n ranges from 1 to 50.

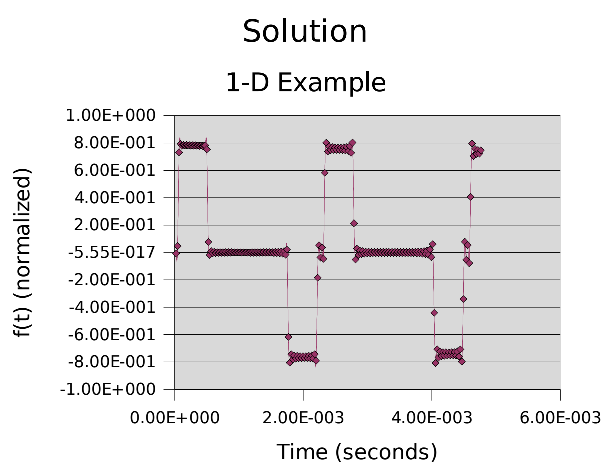
$$\frac{A_n}{\omega_n} = \frac{2uc}{n\pi}\sin\left(\frac{n\pi x''}{a}\right)\sin\left(\frac{n\pi x}{a}\right)$$

Although u and c are normally specified by the problem, for the purposes of this example, we can choose u so that regardless of what c is, we can normalize $\frac{A_n}{\omega_n}$ for the fundamental frequency. For $\left(\frac{x}{a}\right) = \frac{1}{5}$ and $\left(\frac{x''}{a}\right) = \frac{1}{4}$:

$$\frac{A_n}{\omega_n} = \frac{1}{n}\sin\left(\frac{n\pi}{5}\right)\sin\left(\frac{n\pi}{4}\right)$$

Plotting $\frac{A_n}{\omega_n}$ versus ω_n and the first two cycles of $f(t)$ where no filtering has been applied:





In fully 3-D calculations, the frequency spectrum is much more complex. A typical concert hall of say 12,000 cubic meters can have something on the order of three billion modes. The impulse response function is correspondingly more complex, showing the spikes of the direct wave and other waves that are reflected again and again as they arrive at the listening point. As time progresses, the spikes become more and more dense, representing reverberation. Here in this 1-D example, we see merely the effects of the pulses as they pass by again and again after being reflected from the ends of the 1-D medium. Readers familiar with the models of plucked strings may notice that this solution consists of long pauses at zero deflection, hence does not accurately represent a plucked string. The Green's function technique can, however, be used to model plucked strings; this author plans to address the plucked string problem in a separate note.

One may, of course, use this 1-D solution as an impulse response function, convolving it with a different type of waveform to create a different type of attack. Truncated sawtooth, truncated sin, or other short waveforms can be used to create different types of sounds with this same fundamental frequency. One may also use recorded sounds for this convolution, such as those created by striking a drinking glass with a spoon, hitting a desktop with a pen, and so on. This author has used recordings of the pluckings of damped electric guitar strings for this purpose. One can also change the positions of the impulse and the "pickup" to create different types of sounds at this same fundamental frequency. Needless to say, there are countless variations of these types of themes that can be used to create a wealth of sounds with the technique described in this note.

After an instrument is constructed, two different paths could be followed to create a fully 3-D instrument and/or performance: 1) Process each individual solution (note) using a convolution with a stereo impulse response function for a 3-D environment as described in [1], creating an accurate stereo image of each note somewhere in 3-D space; this creates a stereo instrument, perhaps an extended instrument with different notes in different locations; or 2) Assemble an entire "recording" then process the result with the stereo impulse response function, creating a completed 3-D rendering of a performance of the instrument.

References and Notes

[1] David R. Clark. Green's function method for creating accurate stereo sound images. EXE Consulting, P.O. Box 450998, Garland, Texas 75045-0998, July 2004.

[2] Gabriel Barton. *Elements of Green's Functions and Propagation: Potentials, Diffusion, and Waves.* Oxford, 1989. Reprinted 1991.

[3] To fully discuss what is meant by "before any periodic effects become obvious" and "cut the note off in time" would take us well beyond the scope of this note. Suffice it to say that in practice for the case of 1-D, the decay time can be significantly longer than for 3-D where it is limited to about N/2 (where N is the sample rate) unless some scaling is done.

Appendix

A very simple C++ program for producing the data in the plots above follows. Output is to standard out. The sound of this particular impulse response function resembles a banjo at 440 Hz.

```
// One-dimensional pedagogical example for demonstrating application of
//    Green's function formulation.  No windowing or filtering is performed.

// Maximum frequency for spectra is one-half of sample rate
//    (e.g. 44.1 ksps -> 22.050 kHz)

// Dirac delta-function source at xpp = (x'' / length)
// Pickup at x = (x / length)

// WARNING: Be sure to REDIRECT OUTPUT:
//    $ example > example.out

#include <iomanip>
#include <iostream>
#include <cmath>

int main(int argc, char * argv[])
{

// Inputs:

   double sample_rate = 44100.0;
   double freq = 440.0;
   double freq_max = sample_rate / 2.0;
```

```
      int N = (int)(rint(freq_max / freq));
      int T = (int)(rint(freq_max));

      double x = 0.2;
      double xpp = 0.25;

      double pi = 3.141592653589793;
      double * coeff = new double[N+1];
      double * f = new double[T+1];


   // A_n/omega_n:

      for (int n=1; n<=N; n++)   {
         coeff[n] = (1.0/double(n)) * sin(double(n) * pi * xpp) *
            sin(double(n) * pi * x);
         std::cout << "freq = " << freq * n * 2 * pi << "   "
                   << "coeff[" << n << "] = " << coeff[n] << std::endl;
      }

   // f(t) with decay:

      double alpha = log(1e-3) / T;   // t60 for half-period (T)
      for (int t=1; t<=T; t++)   {
         f[t] = 0.0;
         for (int n=1; n<=N; n++)   {
            f[t] = f[t] + coeff[n] *
              sin(double(n) * pi * (2 * freq)  *  double(t)/sample_rate);
         }
         // decay:
         f[t] = f[t] * exp(alpha * double(t));
         std::cout << "time = " << double(t)/sample_rate << "   "
                   << "f[" << t << "] = " << f[t] << std::endl;
      }


      delete[] coeff;
      delete[] f;

      return(0);
}
```